Chapter 1

PARTICLE ACCELERATION IN PULSAR OUTER MAGNETOSPHERES: ELECTRODYNAMICS AND HIGH-ENERGY EMISSIONS

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Abstract

In the last 15 years our knowledge on the high-energy emission from rotation-powered pulsars has considerably improved. The seven known gamma-ray pulsars provide us with essential information on the properties of the particle accelerator — electro-static potential drop along the magnetic field lines. Although the accelerator theory has been studied for over 30 years, the origin of the pulsed gamma-rays is an unsettled question. In this chapter, we review accelerator theories, focusing on the electrodynamics of outer-magnetospheric gap models. Then we present a modern accelerator theory, in which the Poisson equation for the electro-static potential is solved self-consistently with the Boltzmann equations for electrons, positrons, and gamma-rays.

Keywords: gamma-rays: observations, gamma-rays: theory, magnetic fields, pulsars: individual, (Crab pulsar)

1 Introduction

The Energetic Gamma Ray Experiment Telescope (EGRET) on board the Compton Gamma-ray Observatory (CGRO) has detected pulsed signals from at least six rotation-powered pulsars (for the Crab pulsar, Nolan et al. 1993, Fierro et al. 1998; for the Vela

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pulsar, Kanbach et al. 1994, Fierro et al. 1998; for PSR B1706–44, Thompson et al. 1996; for PSR B1951, Ramanamurthy et al. 1995; for Geminga, Mayer-Hasselwander et al. 1994, Fierro et al. 1998; for PSR B1055–52, Thompson et al. 1999). All of them show a double peak in their light curves. At lower energies (below 10 MeV), COMPTEL aboard CGRO also detected pulsation from PSR B1509–58 (Kuiper et al. 1999). In addition to these seven high-confidence pulsar detections above MeV, at least three radio pulsars may have been seen by EGRET (PSR B1046–58, Kaspi et al. 2000; PSR B0656+14, Ramanamurthy et al. 1996; PSR J0218+4232, Kuiper et al. 2000). Since interpreting γ-rays should be less ambiguous compared with reprocessed, non-thermal X-rays, the γ-ray pulsations observed from these objects are particularly important as a direct signature of basic non-thermal processes in pulsar magnetospheres, and potentially should help to discriminate among different emission models.

Spectral energy distributions (SEDs) offers the key to an understanding of the radiation processes. Thompson (1999) compiled a useful set of broad-band SED for the seven high-confidence γ-ray pulsars. The most striking feature of these νFν plots is the flux peak above 0.1 GeV with a turnover at several GeV. Various models (Daugherty & Harding 1982, 1996a,b, hereafter DH82, DH96a,b; Romani and Yadigaroglu 1995, hereafter RY95; Romani 1996, hereafter R96; Zhang & Cheng 1997, hereafter ZC97; Higgins & Henriksen 1997, 1998; Hirotani & Shibata 1999a, b, c, hereafter Papers I, II, III) conclude that these photons are emitted by the electrons or positrons accelerated above 5 TeV via curvature process. Such ultra-relativistic particles could also cause inverse-Compton (IC) scatterings. In particular, for young pulsars, their strong thermal X-rays emitted from the cooling neutron-star surface (Becker& Trümper 1997; Pavlov et al. 2001; Kaminker et al. 2002) efficiently illuminate the gap to be upscattered into several TeV energies. If their magnetospheric infrared (0.1–0.01 eV) photon field is not too strong, a pulsed IC component could be unabsorbed to be detected with future ground-based or space telescopes.

The curvature-emitted photons have the typical energy of a few GeV. Close to the star (typically within a few stellar radii), such γ-rays can be absorbed by the strong magnetic field (> 10¹² G) to materialize as a pair. On the other hand, outside of this strong-field region, pairs are created only by the photon-photon collisions (e.g., between the curvature-GeV photons and surface- or magnetospheric- keV photons). The replenished charges partially screen the original acceleration field, E∥. If the created particles pile up at the boundaries of the potential gap, they will quench the gap eventually. Nevertheless, if the created particles continue to migrate outside of the gap as a part of the global flows of charged particles, a steady charge-deficient region will be maintained. This is the basic idea of a particle acceleration zone in a pulsar magnetosphere.

The pulsar magnetosphere can be divided into two zones: The closed zone filled with a dense plasma corotating with the star, and the open zone in which plasma flows along the open field lines to escape through the light cylinder (left panel of fig. 1). These two zones are separated by the the last-open magnetic field lines, which become parallel to the rotation axis at the light cylinder. Here, the light cylinder is defined as the surface where the corotational speed of a plasma would coincide with the speed of light, c, and hence its distance from the rotation axis is given by Lc ≡ c/Ω. If a plasma flows along the magnetic field line, causality requires that the plasma should migrate outward outside of the light cylinder. In all the pulsar emission models, particle acceleration takes place within
the open zone.

For an aligned rotator (i.e., magnetic inclination angle, $\alpha$, vanishes), open zone occupies the magnetic colatitudes (measured from the magnetic axis) that is less than $\sqrt{r_s/\Omega_{\text{LC}}}$, where $r_s$ represents the neutron-star radius. For an oblique rotator, even though the open-zone polar cap shape is distorted, $\pi(r_s/\Omega_{\text{LC}})$ gives a good estimate of the polar cap area.

On the spinning neutron star surface, from the magnetic pole to the rim of the polar cap, an electro-motive force (EMF) $\approx \Omega^2 B_s r_s^3/c^2 \approx 10^{16.5}$ V is exerted, where $B_s$ refers to the polar-cap magnetic field strength. In this chapter, we assume that both the spin and magnetic axes reside in the same hemisphere; that is, $\Omega \cdot \vec{\mu} > 0$, where $\vec{\Omega}$ represents the rotation vector, and $\vec{\mu}$ the stellar magnetic moment vector. In this case, the strong EMF induces the magnetospheric currents that flow outwards in the lower latitudes and inwards along the magnetic axis (right panel in fig. 1). The return current is formed at large-distances where Poynting flux is converted into kinetic energy of particles or dissipated (Shibata 1997).

Attempts to model the pulsed $\gamma$-ray emissions have concentrated on two scenarios (fig. 1): Polar cap models with emission altitudes of $\sim 10^6$ cm to several neutron star radii over a pulsar polar cap surface (Harding, Tademaru, & Esposito 1978; DH82; DH96a,b; Dermer & Sturmer 1994; Sturmer, Dermer, & Michel 1995; for the slot gap model also see Scharlemann, Arons, & Fawley 1978, hereafter SAF78), and outer gap models with acceleration occurring in the open field zone located near the light cylinder (Cheng, Ho, & Ruderman 1986a,b, hereafter CHR86a,b; Chiang & Romani 1992, 1994; R96; RY95). Both models predict that electrons and positrons are accelerated in a charge depletion region, a potential gap, by the electric field along the magnetic field lines to radiate high-energy $\gamma$-rays via the curvature process. However, it is worth noting that average location of energy loss should take place near the light cylinder so that the rotating neutron star may lose enough angular momentum (Shibata 1995), which indicates the existence of the polar-cap accelerator must affect the electrodynamics in the outer magnetosphere.

It is widely accepted from phenomenological studies that coherent radio pulsations are
probably emitted from the polar-cap accelerator. Moreover, coherent radio pulsations provide the primary channel for pulsar discovery (1627 pulsars as of June 2006, see ATNF pulsar catalogue, http://www.atnf.csiro.au/research/pulsar/psrcat/). However, it is, unfortunately, very difficult to reproduce them by considering plasma collective effects. Therefore, in this chapter, I will focus on incoherent high-energy, non-thermal radiation, which is likely the product of the potential gap in the outer magnetosphere.

## 2 Traditional Outer-gap Model

For a static observer, electric field $\vec{E}$ is obtained from the Poisson equation, $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$, where $\rho$ denotes the real charge density. Since only the electric field component projected along the magnetic field line contributes for particle acceleration, we decouple $\vec{E}$ into the parallel $\vec{E}_\parallel$ and perpendicular $\vec{E}_\perp$ components with respect to the magnetic field line to obtain

$$\vec{\nabla} \cdot \vec{E}_\parallel = 4\pi (\rho - \rho_{\text{GJ}}),$$

where

$$\vec{E}_\parallel = -\vec{\nabla} \Psi \equiv -\vec{\nabla} (A_t + \Omega A_\phi)$$

and (Goldreich & Julian 1969; Mestel 1971)

$$\rho_{\text{GJ}} = \frac{\vec{\nabla} \cdot \vec{E}_\perp}{4\pi} = -\frac{\vec{\nabla} \cdot \vec{B}}{2\pi c} + \frac{(\vec{\nabla} \times \vec{r}) \cdot (\vec{B} \times \vec{B})}{4\pi c};$$

$A_t$ and $A_\phi$ represent the scalar potential and magnetic flux function, respectively, $\vec{B}$ the magnetic field vector at position $\vec{r}$ from the stellar center. For more rigorous derivation of the Poisson equation for arbitrary magnetic field configuration in the curved spacetime, see § 4.1. For a Newtonian dipole magnetic field, $\rho_{\text{GJ}}$ changes sign at the null surface (heavy dashed line in fig. 1) on which $B_z$, the magnetic field component projected along the rotation axis, vanishes. If $\rho$ deviates from $\rho_{\text{GJ}}$ in any region, an electric field is exerted along $\vec{B}$. If the potential drop is sufficient, migratory electrons and/or positrons will be accelerated to radiate $\gamma$-rays via curvature and/or inverse-Compton (IC) processes.

To examine the gap electrodynamics, it is convenient to adopt the so-called ‘magnetic coordinates’ $(s, \theta_s, \varphi_s)$ such that $s$ denotes the distance along the field line, $\theta_s$ and $\varphi_s$ the magnetic colatitude and the magnetic azimuth of the point where the field line intersects the stellar surface; $\theta_s = 0$ and $\varphi_s = 0$ correspond to the magnetic axis and to the plane on which both the rotation and the magnetic axes reside, respectively. If we assume that the gap is thin in the meridional direction in the sense that the $\theta_s$ derivatives dominate the $s$ and $\varphi_s$ derivatives in the left-hand side of equation (1), we obtain (SAF78; see also Muslimov & Tsygan 1992 for the general-relativistic expression)

$$-\frac{1}{r^2} \left( \frac{\partial \theta_s}{\partial \theta} \right)^2 \frac{1}{\theta_s} \frac{\partial}{\partial \theta_s} \left( \theta_s \frac{\partial \Psi}{\partial \theta_s} \right) = -\frac{1}{r^2} \frac{B}{B_s} \frac{1}{\theta_s} \frac{\partial}{\partial \theta_s} \left( \theta_s \frac{\partial \Psi}{\partial \theta_s} \right) = 4\pi (\rho - \rho_{\text{GJ}}).$$

Neglecting the $\theta_s$ dependence of the effective charge $\rho_{\text{eff}} \equiv \rho - \rho_{\text{GJ}}$, we obtain

$$\Psi \approx \frac{\Omega B_s}{c} \left( \frac{\rho}{\Omega B/2\pi c} - \frac{\rho_{\text{GJ}}}{\Omega B_{\text{GJ}}/2\pi c} \right) r_s^2 (\theta_s - \theta_s^{\text{min}})(\theta_s^{\text{max}} - \theta_s),$$

where
where $\theta^\text{max}_s$ and $\theta^\text{min}_s$ give the lower and upper boundary positions, respectively (fig. 2). We should notice here that in traditional outer-gap model, it is assumed $|p| \ll \rho_\text{GJ}$. Thus, $\rho_\text{GJ}/B \propto B_\zeta/B$ essentially determines the $s$ dependence of $\Psi$ and hence $E_\parallel \equiv -\partial_s \Psi$. Since $\rho_\text{GJ}/B$ is roughly proportional to $s - s_0$ (at least near the null surface, $s = s_0$), one obtains a roughly constant $E_\parallel$ in the traditional outer-gap model. The strength of $E_\parallel$ is determined by the Goldreich-Julian charge density at the surface, $\Omega B_\zeta/2\pi c$. Moreover, $E_\parallel(s, \theta_s)$ has a quadratic dependence on $\theta_s$ because of the two zero-potential walls at $\theta_s = \theta^\text{max}_s$ and $\theta^\text{min}_s$. By solving equation (1) on the two-dimensional plane $(s, \theta_s)$, we can confirm later that $\Psi > 0$ holds for $s < s_0$ and $\Psi < 0$ for $s > s_0$ provided that the gap is vacuum (i.e., $\rho = 0$) and geometrically thin (i.e., $\theta^\text{max}_s - \theta^\text{min}_s \ll 1$). Therefore, a vacuum, thin gap starts from the null surface and extends towards the light cylinder. The region $0 < s < s_0$ will be filled with the electrons extracted from the stellar surface, $s = 0$, and hence inactive. Strictly speaking, because of the non-zero contribution of $\rho$ (and also of the two-dimensional effect), the inner boundary $s = s_{\text{in}}$ of such a gap is located slightly inside of $s = s_0$.

In traditional outer-gap models, pulsar high-energy properties have been argued, assuming that the (dimensionless) created current density per magnetic flux tube, $j_{\text{gap}} = -2\pi c \rho(s_{\text{in}})/\Omega B(s_{\text{in}})$ becomes comparable to the typical Goldreich-Julian value, $B_\zeta/B_s$, where $B_\zeta$ represents $B_\zeta$ at the stellar surface. Note that $j_{\text{gap}} \sim B_\zeta/B_s$ is necessary to obtain the observed $\gamma$-ray fluxes and that only electrons (with velocity $-c$) exist at $s = s_{\text{in}}$. However, the non-vanishing $\rho(s_{\text{in}}) = -j_{\text{gap}} \Omega B/2\pi c$ cannot be neglected in the right-hand side of equation (1) at $s = s_0$, where $\rho_\text{GJ}$ vanishes. This fact shows that the traditional thought that $s_{\text{in}} \approx s_0$ breaks down for non-vacuum gaps.

Let us consider the inner boundary position under non-vanishing created charges. In the outer part of the gap, $\rho/B$ tends to be constant because of the decreasing pair creation. Thus, $-\partial_s(\rho_{\text{eff}}/B) \approx \partial_s(\rho_\text{GJ}/B) > 0$ leads to a positive $E_\parallel$ in a transversely thin gap (eq. [5]). In order not to change the sign of $E_\parallel$, $\partial_s E_\parallel \approx 4\pi(\rho - \rho_\text{GJ})$ should be positive in the vicinity of the inner boundary (eq. [1]). Thus, we obtain

$$-\frac{\rho_\text{GJ}}{B} \sim \frac{\Omega}{2\pi c} \frac{B_\zeta}{B} > \frac{-\rho}{B} = j_{\text{gap}} \frac{\Omega}{2\pi c},$$

at $s = s_{\text{in}}$. It follows that the polar cap is the only place for the inner boundary of the ‘outer’ gap to be located, if $j_{\text{gap}}$ is as large as $B_\zeta/B_s$. (On the other hand, if $j_{\text{gap}} \ll B_\zeta/B_s$ holds,
the inner boundary is located near the null surface, where $B_\zeta$ vanishes.) Such a non-vacuum gap must extend from the polar cap surface (not from the null surface as traditionally assumed) to the outer magnetosphere. We can therefore conclude that the original vacuum solution obtained by CHR86a,b cannot be applied to a non-vacuum gap, which is necessary to explain the observed $\gamma$-ray fluxes. To construct a self-consistent model, we have to solve equation (1) together with the Boltzmann equations for particles and $\gamma$-rays. This scheme was first proposed by Beskin et al. (1992, hereafter BIP92) for a magnetosphere of a super-massive black hole (SMBH), and developed by Hirotani and Okamoto (1997, hereafter Paper 0), then applied to pulsar magnetospheres (Papers I, II, III). In what follows, we review this method of approach.

For convenience, we summarize accelerator models in table 1. The three-dimensional model by RY95, R96, and Cheng et al. (2000, hereafter CRZ00) bases on CHR86a,b. For comparison, we also present representative inner-gap models (Fawley et al. 1977, hereafter FAS77; SAF78; DH96a, b) in the table.

### 3 1-Dimensional Analysis of Gap Electrodynamics

A good place to start is the one-dimensional consideration of gap properties along the magnetic field lines. Neglecting relativistic effects, and assuming that typical transfield thickness, $W_\perp$, is greater than or comparable to the longitudinal width, $W_\parallel$, we can reduce the Poisson equation for non-corotational potential $\Psi$ into the one-dimensional form

$$-\nabla^2\Psi \approx -\frac{d^2\Psi}{ds^2} + \frac{\Psi}{W_\perp^2} = 4\pi \left[ \rho(s) + \frac{\Omega B_\zeta(s)}{2\pi c} \right].$$

(7)

The real charge density can be computed by

$$\rho = e \int_{1}^{\infty} (N_+ - N_-)d\Gamma,$$

(8)

where $N_+(s, \Gamma)$ and $N_-(s, \Gamma)$ refer to the positronic and electronic distribution functions, respectively.

### 3.1 Particle Boltzmann Equations

In general, the distribution functions $N_\pm = N_\pm(t, \vec{x}, \vec{p})$ obey the following Boltzmann equations:

$$\frac{\partial N_\pm}{\partial t} + \frac{\vec{p}}{m_e\Gamma} \cdot \vec{\nabla} N_\pm + F_{\text{ext}}^- \cdot \frac{\partial N_\pm}{\partial \vec{p}} = S_\pm(t, \vec{x}, \vec{p}),$$

(9)

where $\vec{p}$ refers to the particle momentum, $F_{\text{ext}}^-$ the external forces acting on particles, and $S_\pm$ the collision terms. In this chapter, $F_{\text{ext}}^-$ consists of the Lorentz and the curvature radiation reaction forces. Since the magnetic field is much less than the critical value ($4.41 \times 10^{13}$ G), quantum effects can be neglected in the outer magnetosphere. As a result, curvature radiation takes place continuously and can be regarded as an external force acting on a particle. If we instead put the collision term associated with the curvature process in the right-hand side, the energy transfer in each collision would be too small to be resolved by the energy
sphere (on the corotating frame) and impose
\[ \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} = 0. \] (10)

Collision terms are expressed as
\[
S_{\pm}(s, \theta_s, \phi_s, \Gamma, \chi) = - \int_{\Gamma > \Gamma_s}^1 d\mu c \int dE_\gamma \eta IC^\gamma(E_\gamma, \Gamma, \mu c)n_{\pm}(s, \theta_s, \phi_s, \Gamma, \chi)
+ \int_{\Gamma < \Gamma_s}^1 d\mu c \int d\Gamma_s \eta IC^c(\Gamma_s, \Gamma, \mu c)n_{\pm}(s, \theta_s, \phi_s, \Gamma, \chi)
+ \int d\mu c \int dE_\gamma \left[ \frac{\partial \eta IC^\gamma(E_\gamma, \Gamma, \mu c)}{\partial \Gamma} + \frac{\partial \eta IC^c(\Gamma_s, \Gamma, \mu c)}{\partial \Gamma} \right] \frac{B_s}{B} g_{\pm}(r, E_\gamma, \tilde{k}),
\] (11)

where \( \mu c \) refers to the cosine of the collision angle between the particle and the soft photon for inverse-Compton scatterings (ICS), between the \( \gamma \)-ray and the soft photon for two-photon pair creation, and between the \( \gamma \)-ray and the local magnetic field line for one-photon pair creation; \( \chi \) denotes the pitch angle of the particle. The function \( g \) represents the \( \gamma \)-ray distribution function divided by \( \Omega B_s/(2\pi c e) \) at energy \( E_\gamma \), momentum \( \tilde{k} \) and position \( \tilde{r} \). In the third line, \( \mu c \) is computed from the photon propagation direction, \( \tilde{k}/|k| \). Since pair annihilation is negligible, we do not include this effect in equation (11). For the explicit expressions of the re-distribution functions, \( \eta IC^\gamma, \eta IC^c, \eta IC^\gamma; \) and \( \eta IC^c \), see § 4.2 (see also Hirotani 2006a, hereafter Paper XI).

For the sake of one-dimensional analysis, in this section, we suppress the dependence on \( \theta_s, \phi_s, \) and \( \chi \) in equations (9) and (11) to elucidate basic features of gap electrodynamics in the one-dimensional approximation. To suppress the dependence of \( g \) on \( \tilde{k}/|k| \), we also assume that \( \gamma \)-rays propagate parallel to the local magnetic field line and adopt \( \tilde{k} \parallel \tilde{B} \).

Imposing stationary condition (10), we then obtain
\[
\pm c \frac{\partial n_{\pm}}{\partial s} + \frac{1}{m_e c} \left( eE_\parallel - \frac{P_{sc}}{c} \right) \frac{\partial n_{\pm}}{\partial \Gamma} = S_{\pm}.
\] (12)

where
\[
S_{\pm}(\Gamma) = - \int_{E_\gamma < \Gamma} dE_\gamma \eta IC^\gamma(E_\gamma, \Gamma, \mu c)n_{\pm}(s, \Gamma)
+ \int_{E_\gamma > \Gamma} dE_\gamma \eta IC^c(\Gamma_s, \Gamma, \mu c)n_{\pm}(s, \Gamma_s) + \int dE_\gamma \frac{\partial \eta IC^\gamma(\Gamma, \Gamma, \mu c)}{\partial \Gamma} \frac{B_s}{B} g_{\pm}(r, E_\gamma, \tilde{k}).
\] (13)

Note that \( \mu c \) is determined at each position \( s \). In the first and second terms in the right-hand side, we assume the surface blackbody component as the source of the photons to be up-scattered and adopt uni-directional approximation to their specific intensity. In the third line, \( \tilde{k} \parallel \tilde{B} \) determines \( \mu c \) at each point.
3.2 Gamma-ray Boltzmann Equations

In general, the $\gamma$-ray distribution function $g$ obeys the Boltzmann equation,

$$\frac{\partial g}{\partial t} + c \frac{k}{|k|} \cdot \nabla g(t, \vec{r}, \vec{k}) = S_\gamma(t, \vec{r}, \vec{k}),$$  \hspace{1cm} (14)

where $|k|^2 = -k^2 k_j$; $S_\gamma$ is given by

$$S_\gamma = - \int_{-1}^{1} d\mu_c \int_{1}^{\infty} d\Gamma \frac{\partial}{\partial \Gamma} \left[ \eta_{\gamma\gamma}(\vec{r}, \Gamma, \mu_c) + \eta_{\gamma\beta}(\vec{r}, \Gamma, \mu_c) \right] \cdot g(\vec{r}, \vec{k}),$$

$$+ \int_{-1}^{1} d\mu_c \int_{1}^{\infty} d\Gamma \eta_{\gamma\gamma}(s, \Gamma, \mu_c) \frac{B}{B_s} n_\pm(s, \theta_s, \varphi_s, \Gamma, \chi)$$

$$+ \int_{0}^{\pi} d\chi \int_{1}^{\infty} d\Gamma \eta_{\gamma\gamma}(s, \Gamma, \chi) \frac{B}{B_s} n_\pm(s, \theta_s, \varphi_s, \Gamma, \chi),$$ \hspace{1cm} (15)

where $\eta_{\gamma\gamma}$ is the synchro-curvature radiation rate [s$^{-1}$] into the energy interval between $E_\gamma$ and $E_\gamma + dE_\gamma$ by a particle migrating with Lorentz factor $\Gamma$ and pitch angle $\chi$. For explicit expression, see Cheng and Zhang (1996).

Imposing the stationary condition (10), and denoting $g_\pm$ as the $g$ associated with the $\gamma$-rays propagating in $\pm \vec{B}$ directions, we obtain the following Boltzmann equations:

$$\pm c \frac{dg_{\pm}}{ds} = S_{\gamma, \pm}(s, E_\gamma),$$ \hspace{1cm} (16)

where

$$S_{\gamma, \pm} = - \int_{1}^{\infty} d\Gamma \frac{\partial \eta_{\gamma\gamma}(s, \Gamma, \mu_c)}{\partial \Gamma} \cdot g_\pm(s, E_\gamma)$$

$$+ \int_{1}^{\infty} d\Gamma \eta_{\gamma\gamma}(E_\gamma, \Gamma, \mu_c) \frac{B}{B_s} n_\pm(s, \Gamma) + \int_{1}^{\infty} d\Gamma \eta_{\gamma\gamma}(E_\gamma, \Gamma, \chi) \frac{B}{B_s} n_\pm(s, \Gamma),$$ \hspace{1cm} (17)

3.3 Boundary Conditions

The Poisson equation and the $\gamma$-ray Boltzmann equations are ordinary differential equations, which can be straightforwardly solved by a simple discretization. On the other hand, the hyperbolic-type partial differential equations (12) are solved by the Cubic Interpolated Propagation (CIP) method (e.g., Yabe & Aoki 1991, Yabe, Xiao, & Utsumi 2001). We represent $n_\pm$ at $\Gamma = \beta_l$ ($l = 1, 2, 3, \ldots, m$) with $n_{\pm, l}$, where $\beta_l$ is a linearly gridded Lorentz factor variable.

At the inner (starward) boundary ($s = s_{in}$), we impose (Hirotani & Shibata 2001a, b, hereafter Papers VII, VIII)

$$E_{\parallel}(s_{in}) = 0, \quad \Psi(s_{in}) = 0,$$ \hspace{1cm} (18)

$$g_{\gamma}^{l}(s_{in}) = 0 \quad (l = l_0 + 1, \ldots, -1, 0, 1, 2, \ldots, m),$$ \hspace{1cm} (19)

$$n_{+, l}(s_{in}) = \begin{cases} \hat{n}_{in}, & \text{for } l = 1 \\ 0, & \text{for } l = 2, 3, \ldots, m \end{cases}$$ \hspace{1cm} (20)
where the dimensionless positronic injection current across the inner boundary \( s = s_{\text{in}} \) is denoted as \( j_{\text{in}} \). In this section, we adopt \( \beta_0 = 10^5 \). Current conservation gives another constraint

\[
\sum_I n_{-I}(s_{\text{in}}) = j_{\text{out}} - j_{\text{in}}. \tag{21}
\]

At the outer boundary \( (s = s_{\text{out}}) \), we impose

\[
E_{\parallel}(s_{\text{out}}) = 0, \tag{22}
\]

\[
g_{-i}(s_{\text{out}}) = 0 \quad (i = l_0 + 1, \ldots, -1, 0, 1, 2, \ldots, m), \tag{23}
\]

\[
n_{-I}(s_{\text{out}}) = \begin{cases} 
    j_{\text{out}}, & \text{for } l = 1 \\
    0, & \text{for } l = 2, 3, \ldots, m 
\end{cases} \tag{24}
\]

The current density created in the gap per unit flux tube can be expressed as

\[
j_{\text{gap}} = j_{\text{tot}} - j_{\text{in}} - j_{\text{out}}. \tag{25}
\]

We adopt \( j_{\text{gap}} \), \( j_{\text{in}} \), and \( j_{\text{out}} \) as the free parameters.

We have totally \( 2m + 2(m - l_0 + 1) + 4 \) boundary conditions (18)–(24) for \( 2m + 2(m - l_0 + 1) + 2 \) unknown functions \( n_{\pm I} \), \( g_{\pm i} \), \( \Psi \), and \( E_{\parallel} \). Thus two extra boundary conditions must be compensated by making the positions of the boundaries \( s_{\text{in}} \) and \( s_{\text{out}} \) be free. The two free boundaries appear because \( E_{\parallel} = 0 \) is imposed at both the boundaries and because \( j_{\text{gap}} \) is externally imposed. In other words, the gap boundaries \( (s_{\text{in}} \) and \( s_{\text{out}} \) shift, if \( j_{\text{in}} \) and/or \( j_{\text{out}} \) varies.

### 3.4 Mono-energetic Approximation: Acceleration Electric Field

To aid in grasping the basic features, we first adopt the mono-energetic approximation in which \( \Gamma \) dependence of \( n_{\pm}(s, \Gamma) \) is suppressed, assuming that all the particles have the Lorentz factor that is given by the balance between the electro-static acceleration and the curvature radiation-reaction force. Under this simplification, all the basic equations (7), (12), and (16) become ordinary differential equations. Note that \( m = 1 \) is adopted (see § 3.3).

Let us begin with browsing some examples of \( E_{\parallel}(s) \) solved for several representative values of the created current density per magnetic flux tube, \( j_{\text{gap}} \). In figure 3, we present \( E_{\parallel}(s) \) and the equilibrium Lorentz factor, which is obtained by equating \( P_{\text{SC}}/c \) with \( eE_{\parallel} \), for \( j_{\text{gap}} = 0.10, 0.20, 0.218 \) with solid, dashed, and dotted curves, respectively (Paper III). Other parameters are chosen to be \( \Omega = 100 \text{rad s}^{-1}, \mu = 10^{30} \text{G cm}^3, W_{\perp} = 10^8 \text{ cm}, kT = 75 \text{ eV} \) (surface blackbody temperature), and \( \alpha_i = 0^\circ \) (i.e., aligned rotator). In this figure, abscissa denotes the distance along the field line, \( x = s - s_0 \).

For a small \( j_{\text{gap}} \), the gap becomes nearly vacuum. Moreover, unless the gap width, \( W_{\perp} \equiv s_{\text{out}} - s_{\text{in}} \), exceeds \( W_{\perp} \), the term \( \Psi/W_{\perp}^2 \), which describes the 2-dimensional screening effect, is not important. As a result, equation (7) gives approximately a quadratic solution,

\[
E_{\parallel}(s) = E_{\parallel}(s_0) - \frac{\Omega}{c} \left( \frac{\partial B_{\perp}}{\partial s} \right)_0 (s - s_0)^2. \tag{26}
\]
The solid line in the left panel of figure 3 (i.e., small \( j_{\text{gap}} \) case) approximately represents such a quadratic solution. Since the trans-field thickness \( (= 10^8 \text{cm}) \) is comparable with the longitudinal gap width in the present case, the two-dimensional effect contributes to shift the gap outwards. In another word, without the \( \Psi/W_\perp^2 \) term, the solid curve would become symmetric with respect to \( x = s - s_0 = 0 \).

It is worth examining the case when \( W_\perp \) becomes smaller. In figure 4, we present \( E_k(s) \) for three discrete values of \( W_\perp \) (Paper I). Other quantities are fixed as \( j_{\text{gap}} = 0.1 \), \( \Omega = 100 \text{rad s}^{-1} \) and \( \alpha_i = 0^\circ \). Instead of specifying \( \mu \) and \( kT \), we adopt \( B = 10^5 \text{G} \) at \( s = s_0 \) and \( L = 10^{33} \text{ergs s}^{-1} \) for the surface blackbody luminosity. It follows that \( E_k(s) \) symmetrically distribute with respect to the null surface when \( W_\perp \) is small and that the gap shift outwards with decreasing \( W_\perp \) because of the two-dimensional screening effect. It suggests that a vacuum, transversely thin (i.e., \( j_{\text{gap}} \ll 1 \) and \( W_\perp \ll W_k \) gap starts from the null surface to extend towards the light cylinder with a nearly constant \( E_k \). It will be confirmed by a full two-dimensional analysis in § 4.5.1.

Let us examine how \( E_k(s) \) changes with particle injection. In figure 5, we present the case for PSR B1055–52 under one-dimensional and mono-energetic approximations (Hirotoni & Shibata 2002, hereafter Paper IX). We adopt \( j_{\text{gap}} = 0.01 \) and \( j_{\text{out}} = 0 \). The solid, dashed, dash-dotted, and dotted curves in the left panel correspond to \( j_{\text{in}} = 0, 0.25, 0.5, \) and 0.585, respectively. It follows that the gap is located near the null surface if there is no particle injection but shifts outwards as \( j_{\text{in}} \) increases. The right panel represents \( n_+(s) \) (thick solid curve), \( n_-(s) \) (thin solid), and \( j_{\text{tot}} = j_{\text{gap}} = n_+(s) + n_-(s) \) (dashed), for \( j_{\text{in}} = j_{\text{out}} = 0 \) and \( j_{\text{gap}} = 0.01 \), which corresponds to the solution represented by the solid curve in the left panel. It follows that the pair creation mainly takes place in the inner region of the gap. This is because most of the pairs are created by the (more or less head-on) collisions between inward-directed \( \gamma \)-rays and surface X-rays. Since created pairs at \( s \) is proportional to the number of \( \gamma \)-rays emitted at larger \( s \) (via curvature process), \( n_- \) (thin curve) increases roughly exponentially with decreasing \( s \). Note that the current one-dimensional treatment
enhances pair creation, because $\gamma$-rays do not escape into the convex side due to the field-line curvature.

3.5 Mono-energetic Approximation: Gap Position vs. Injected Current

In this section, we analytically investigate why the gap position shifts outwards with increasing particle injection across the inner boundary, adopting the mono-energetic approximation.

3.5.1 Particle Continuity Equations

Integrating equation (9) over the momentum space, and assuming that $N$ vanishes rapidly enough at $p_i \to \pm \infty$ ($i = 1, 2, 3$), we obtain

$$\frac{\partial \tilde{N}}{\partial t} + \tilde{\nabla} \cdot \langle \tilde{v} \rangle \tilde{N} = \tilde{S}(t, \tilde{x}),$$  \hspace{1cm} (27)

where the particle number density $\tilde{N}$ and the averaged particle velocity $\langle \tilde{v} \rangle$ are defined by

$$\tilde{N}(t, \tilde{x}) \equiv \int_{-\infty}^{\infty} N(t, \tilde{x}, \tilde{p}) d^3 \tilde{p}, \quad \langle \tilde{v} \rangle \equiv \frac{\int_{-\infty}^{\infty} \tilde{v} N d^3 \tilde{p}}{\int_{-\infty}^{\infty} N d^3 \tilde{p}}.$$

Since the IC scatterings and the synchro-curvature process conserve the particle number,

$$\tilde{S}(t, \tilde{x}) \equiv \int_{-\infty}^{\infty} S(t, \tilde{x}, \tilde{p}) d^3 \tilde{p}$$

(29)
consists of pair creation and annihilation terms. For a typical pulsar magnetosphere, annihilation is negligibly small compared with creation. Therefore, we obtain

$$\tilde{S}(t, \bar{x}) = \frac{1}{c} \int_0^\infty dE_\gamma \left[ \eta_\gamma(\bar{x}, E_\gamma, \mu_+) G_+ + \eta_\gamma(\bar{x}, E_\gamma, \mu_-) G_- \right],$$  \hspace{1cm} (30)

where $G_+ (t, \bar{x}, E_\gamma)$ and $G_- (t, \bar{x}, E_\gamma)$ designate the distribution functions of outward- and inward-directed $\gamma$-ray photons, respectively. The pair-creation redistribution functions are defined by

$$\eta_\gamma(\bar{x}, E_\gamma, \mu) = c \int_{-1}^1 d\mu (1 - \mu) \int_{E_{\text{th}}}^\infty dE_x \frac{dN_x}{dE_x d\mu} \sigma_p,$$  \hspace{1cm} (31)

where $\sigma_p(E_\gamma, E_x, \mu)$ represents the pair-creation cross section, and

$$E_{\text{th}} \equiv \frac{2}{1 - \mu} \frac{(m_e c^2)^2}{E_\gamma};$$  \hspace{1cm} (32)

cos$^{-1} \mu_+$ (or cos$^{-1} \mu_-$) is the collision angle between the X-rays and the outwardly (or inwardly) propagating $\gamma$-rays.

Imposing a stationary condition (10), and utilizing $\nabla \cdot B = 0$, we obtain from equation (27),

$$\pm B \frac{\partial}{\partial s} \left( \frac{\tilde{N}_\pm}{B} \right) = \frac{1}{\lambda_\gamma} \int_0^\infty dE_\gamma (G_+ + G_-),$$  \hspace{1cm} (33)
The pair-creation mean free path $\lambda_p(s)$ is defined by

$$\frac{1}{\lambda_p} \equiv \frac{\int_0^\infty \left[ \eta_p(s, E_\gamma, \mu_+) G_+ + \eta_p(s, E_\gamma, \mu_-) G_- \right] dE_\gamma}{c \int_0^\infty (G_+(s, E_\gamma) + G_-(s, E_\gamma)) dE_\gamma}. \quad (34)$$

Since the number of created positrons is always equal to that of created electrons, the right-hand side of equation (33) is common for $\tilde{N}_+$ and $\tilde{N}_-$.

In terms of $G_\pm \equiv (\Omega B_s/2\pi c e) g_\pm$ and $N_\pm \equiv (\Omega B/2\pi c e) n_\pm$, equation (16) gives

$$\pm c \frac{dG_\pm}{ds} \approx \int_1^\infty \eta_{SC} N_\pm d\Gamma, \quad (35)$$

where absorption and ICS are neglected in $S_{\gamma \pm}$. Integrating this equation over $E_\gamma$, combining with equations (33), and assuming $\partial_s(\lambda_p \cos \Phi) = 0$, we obtain

$$\pm \frac{d^2}{ds^2} \left( \frac{N_\pm}{B} \right) = \frac{1}{\lambda_p c} \frac{N_+ - N_-}{B} \int_0^\infty \eta_{SC} dE_\gamma. \quad (36)$$

### 3.5.2 Real Charge Density in the Gap

One combination of the two independent equations (36) yields the current conservation law; that is, the total current density per magnetic flux tube,

$$j_{tot} = \frac{2\pi c e}{\Omega} \frac{\tilde{N}_+(s) + \tilde{N}_-(s)}{B(s)} \quad (37)$$

is conserved along the field lines. (Note that it can be derived directly from eq. [33].) Another combination gives

$$\frac{d^2}{ds^2} \left( \frac{\tilde{N}_+ - \tilde{N}_-}{B} \right) = \frac{2}{W||} \frac{N_+ \tilde{N}_+ - \tilde{N}_-}{\lambda_p B}, \quad (38)$$

where

$$N_\gamma \equiv \frac{W}{c} \int_0^\infty \eta_{SC}(s, \Gamma, E_\gamma) dE_\gamma \quad (39)$$

refers to the expectation value of the number of $\gamma$-rays emitted by a single particle that runs the gap width, $W|| = s_{out} - s_{in}$. Lorentz factor appearing in $\eta_{SC}$ should be evaluated at each position $s$.

Exactly speaking, $\lambda_p$ depends on $G_+$ and $G_-; \text{ thus, the } \gamma\text{-ray distribution functions are not eliminated in equation (36). Nevertheless, for analytic (and qualitative) discussion of the gap position, we may ignore such details and adopt equation (38).}$

A typical $\gamma$-ray propagates the length $W||/2$ within the gap that is transversely thick. Thus, so that a stationary pair-creation cascade may be maintained, the optical depth, $W||/(2\lambda_p)$, must equal the expectation value for a $\gamma$-ray to materialize with the gap, $N^{-1}_\gamma$. We thus obtain the following condition: $W||/2 = \lambda_p/N_\gamma$. This relation holds for a self-sustaining gap in which all the particles are supplied by the pair creation. If there is an external particle injection, the injected particles also contribute for the $\gamma$-ray emission. As
a result, a stationary gap can be maintain with a smaller width compared to the case of no particle injection. Taking account of such injected particles, we can constrain the half gap width as

$$W_0 = \frac{\lambda_p}{N_N} \frac{j_{\text{gap}}}{j_{\text{tot}}},$$

(40)

where $j_{\text{gap}}$ and $j_{\text{tot}}$ refer to the created and total current densities per unit magnetic flux tube. Equation (40) is automatically satisfied if we solve the set of Maxwell and stationary Boltzmann equations. Here, $j_{\text{gap}}$ is related with the particle injection rate across the boundaries as follows:

$$W_2 p c e j_{\text{gap}} = \tilde{N}_+ (s_{\text{out}}) - \tilde{N}_- (s_{\text{in}}) = \frac{\Omega}{2\pi c e} \frac{j_{\text{gap}}}{j_{\text{tot}}} = \frac{\Omega}{2\pi c e} \frac{\tilde{N}_- (s_{\text{out}}) - \tilde{N}_- (s_{\text{in}})}{B (s_{\text{in}}) - B (s_{\text{out}})}.$$

(41)

With the aid of identity (40), we can rewrite equation (38) into the form

$$\frac{d^2}{ds^2} \left( \frac{\tilde{N}_+ - \tilde{N}_-}{B} \right) = \frac{4 j_{\text{gap}}}{j_{\text{tot}} W_0^2} \frac{\tilde{N}_+ - \tilde{N}_-}{B}.$$

(42)

To solve this differential equation, we impose the following two boundary conditions:

$$c e \frac{\tilde{N}_+ (s_{\text{in}})}{B (s_{\text{in}})} = \frac{\Omega}{2\pi c e} \frac{j_{\text{gap}} - j_{\text{in}} + j_{\text{out}}}{j_{\text{tot}} W_0^2} \frac{\tilde{N}_- (s_{\text{out}})}{B (s_{\text{out}})} = \frac{\Omega}{2\pi c e} \frac{j_{\text{gap}} + j_{\text{in}} - j_{\text{out}}}{j_{\text{tot}} W_0^2}.$$

(43)

Then, equations (41), (43) give

$$\frac{\tilde{N}_+ - \tilde{N}_-}{B} = -\frac{\Omega}{2\pi c e} (j_{\text{gap}} - j_{\text{in}} + j_{\text{out}}).$$

(44)

at $s = s_{\text{in}}$, and

$$\frac{\tilde{N}_+ - \tilde{N}_-}{B} = \frac{\Omega}{2\pi c e} (j_{\text{gap}} + j_{\text{in}} - j_{\text{out}}).$$

(45)

at $s = s_{\text{out}}$. Under boundary conditions (44) and (45), equation (42) is solved as

$$\frac{\tilde{N}_+ - \tilde{N}_-}{B} = \frac{\Omega}{2\pi c e} \left[ \frac{\sinh (j_{\text{gap}} s - s_{\text{cnt}} / W_0 / 2)}{j_{\text{tot}} \sinh (j_{\text{gap}} / j_{\text{tot}})} + \frac{(j_{\text{in}} - j_{\text{out}}) \cosh (j_{\text{gap}} s - s_{\text{cnt}} / W_0 / 2)}{j_{\text{tot}} \cosh (j_{\text{gap}} / j_{\text{tot}})} \right],$$

(46)

where the gap center position is defined by

$$s_{\text{cnt}} = \frac{s_{\text{in}} + s_{\text{out}}}{2}.$$

(47)

Note that $e (\tilde{N}_+ - \tilde{N}_-) = \rho$ represents the real charge density, which appears in the Poisson equation. Thus, substituting equation (46) into (1), we obtain

$$-\nabla^2 \psi = \frac{2 B \Omega}{c} \left[ j_{\text{gap}} f_{\text{odd}} \frac{(s - s_{\text{cnt}})}{W_0 / 2} + (j_{\text{in}} - j_{\text{out}}) f_{\text{even}} \frac{(s - s_{\text{cnt}})}{W_0 / 2} + \frac{B_0}{B} \right],$$

(48)

where

$$f_{\text{odd}} (x) \equiv \frac{\sinh (x \sqrt{j_{\text{gap}} / j_{\text{tot}}})}{\sinh (\sqrt{j_{\text{gap}} / j_{\text{tot}}})}, \quad f_{\text{even}} (x) \equiv \frac{\cosh (x \sqrt{j_{\text{gap}} / j_{\text{tot}}})}{\cosh (\sqrt{j_{\text{gap}} / j_{\text{tot}}})}.$$
At the inner boundary, \( s = s_{\text{in}} \), \( (s - s_{\text{cnt}})/(W_\parallel/2) = -1 \) holds; therefore, we obtain
\[
-\nabla^2 \Psi \approx \frac{\partial E_\parallel}{\partial s} = \frac{2B\Omega}{c} \left( -j_{\text{gap}} + j_{\text{in}} - j_{\text{out}} + \frac{B_\xi}{B} \right). 
\] (50)

In the CHR picture, it is assumed that there is no particle injection across either of the boundaries (i.e., \( j_{\text{in}} = j_{\text{out}} = 0 \)) and that the current density associated with the created particles becomes of the order of the typical Goldreich-Julian value (i.e., \( j_{\text{gap}} \sim 1 \)). However, it results in a negative \( E_\parallel \) in the vicinity of the inner boundary, and hence in the reversal of \( E_\parallel \) sign. In another word, for a transversely thin gap to have a positive \( E_\parallel \) in the entire region, the right-hand side of equation (48) should be negative in almost entire region but should be \textit{positive} in the vicinity of the inner boundary. It follows that the inner boundary of an ‘outer gap’ should be located close to the star, where \( B_\xi \sim B \) holds. This conclusion is consistent with what was obtained at the end of § 2.

### 3.5.3 Gap Position vs. Particle Injection

To examine the Poisson equation (48) analytically, we assume that the transfield thickness of the gap is greater than \( W_\parallel \) and replace \( \nabla^2 \Psi \) with \( d^2\Psi/ds^2 \). Furthermore, we neglect the current created in the gap and put \( j_{\text{gap}} \sim 0 \).

First, consider the case when particles are injected across neither of the boundaries (i.e., \( j_{\text{in}} = j_{\text{out}} = 0 \)). It follows that the derivative of the \( E_\parallel \) vanishes at the null surface, where \( B_\xi \) vanishes. We may notice that \( -d^2\Psi/ds^2 = dE_\parallel/ds \) is positive at the inner part of the gap and becomes negative at the outer part. The acceleration field is screened out at the boundaries by virtue of the spatial distribution of the local Goldreich-Julian charge density, \( \rho_{\text{GJ}} \). Therefore, we can conclude that the gap is located (or centers) around the null surface, if there is no particle injection from outside.

Secondly, consider the case when particles are injected across the inner boundary at \( s = s_{\text{in}} \) (or in general, when \( j_{\text{in}} - j_{\text{out}} > 0 \) holds). Since the function \( f_{\text{even}} \) is positive at arbitrary \( s \), the gap center is located at a place where \( B_\xi \) is negative, that is, outside of the null surface. In particular, when \( j_{\text{in}} - j_{\text{out}} \sim 1 \) holds, \( dE_\parallel/ds \) vanishes at the place where \( B_\xi \sim -B \) holds. In a vacuum, static dipole field, \( B_\xi \sim -B \) is realized along the last-open field line near to the light cylinder. Therefore, the gap should be located close to the light cylinder, if the injected particle flux across the inner boundary approaches the typical Goldreich-Julian value. We may notice here that \( f_{\text{even}} \) is less than unity, because \( |s - s_{\text{cnt}}| \) does not exceed \( W_\parallel/2 \).

Thirdly and finally, consider the case when \( j_{\text{in}} - j_{\text{out}} \sim -1 \) holds. In this case, \( dE_\parallel/ds \) vanishes at the place where \( B_\xi \sim B \). Therefore, an ‘outer’ gap should be located in the polar cap, if a Goldreich-Julian particle flux is injected across the outer boundary.

### 3.6 Energy Dependence of Particle Distribution Functions

The analytic conclusions derived in the foregoing section can be confirmed numerically discarding the mono-energetic approximation. To this aim, we apply the one-dimensional scheme described in §§ 3.1–3.3 to the Vela pulsar parameters (Paper X).

To compare the effects of particle injection, we present \( E_\parallel(s) \) for the three cases of \( j_{\text{in}} = 0 \) (solid), 0.25 (dashed), and 0.50 (dash-dotted) in figure 6. The magnetic inclination
Figure 6: Results for the Vela pulsar parameter when \( \alpha_i = 75^\circ \), \( j_{\text{gap}} = 4.6 \times 10^{-5} \), and \( j_{\text{out}} = 0 \). The abscissa designates the distance along the last-open field line normalized by the light cylinder radius. Left: \( E_k(s) \) for \( j_{\text{in}} = 0 \) (solid), 0.25 (dashed), and 0.5 (dash-dotted). Right: \( E_k(s) \) (dashed), the equilibrium Lorentz factor (dotted), and positronic characteristics (solid) for \( j_{\text{in}} = 0.25 \) (i.e., the dashed curve case in the right panel). From Paper X.

is chosen to be \( \alpha_i = 75^\circ \). We adopt \( j_{\text{out}} = 0 \) throughout this chapter, unless its value is explicitly specified. The abscissa denotes the distance along the last-open field line normalized by \( \varpi_{\text{LC}} \). As the solid line shows, \( E_k \) is located around the null surface when there is no particle injection across either of the boundaries. Moreover, \( E_k \) varies quadratically, because the Goldreich-Julian charge density deviates from zero nearly linearly near to the null surface. As the dashed and dash-dotted lines indicate, the gap shifts outwards as \( j_{\text{in}} \) increases. When \( j_{\text{in}} = 0.5 \) for instance, the gap is located on the half way between the null surface and the light cylinder. This result is consistent with what is obtained under mono-energetic approximation (Papers VII–IX).

In the right panel, we present the characteristics of partial differential equation (12) for positrons by solid lines, together with \( E_k(s) \) when \( j_{\text{in}} = 0.25 \) and \( \alpha_i = 75^\circ \) (i.e., the dashed line in the right panel). We also superpose the equilibrium Lorentz factor that would be obtained if we assumed the balance between the curvature radiation reaction and the electrostatic acceleration, as the dotted line. It follows that the particles are not saturated at the equilibrium Lorentz factor in most portions of the gap.

In the outer part of the gap where \( E_k \) is decreasing, characteristics begin to concentrate; as a result, the energy distribution of outwardly propagating particles forms a ‘shock’ in the Lorentz factor direction. However, the particle Lorentz factors do not match the equilibrium value (dotted line). For example, near the outer boundary, the particles have larger Lorentz factors compared with the equilibrium value, because the curvature cooling scale is longer than the gap width. Thus, we must discard the mono-energetic approximation that all the particles migrate at the equilibrium Lorentz factor. We instead have to solve the energy dependence of the particle distribution functions explicitly.
In figure 7, we present the energy distribution of particles at several representative points along the field line. At the inner boundary \( s = 0.184 \sigma_{LC} \), particles are injected with Lorentz factors typically less than \( 4 \times 10^6 \) as indicated by the solid line. Particles migrate along the characteristics in the phase space and gradually form a ‘shock’ as the dashed line (at \( s = 0.205 \sigma_{LC} \)) indicates, and attains maximum Lorentz factor at \( s = 0.228 \sigma_{LC} \) as the dash-dotted line indicates. Then they begin to be decelerated gradually and escape from the gap with large Lorentz factors \( \sim 2.5 \times 10^7 \) (dotted line) at the outer boundary, \( s = s_{\text{out}} = 0.241 \sigma_{LC} \).

So far, we have investigate several basic features of gap electrodynamics in one-dimensional approximation. However, in a realistic pulsar magnetosphere, \( \gamma \)-rays propagate in the convex side to deviate from the last-open field line; thus, pair creation rate increases at higher latitudes (i.e., away from the last-open field line). In the next section, we consider the trans-field structure by solving the basic equations on the two-dimensional plane.

4 2-Dimensional Analysis of Gap Electrodynamics

In this section, we recover \( \theta \) dependence in equation (1), \( \theta_s \) and \( \chi \) dependence in equation (9), and \( \theta_s \) and \( k^q/k' \) dependence in equation (14). We still suppress toroidal variables \( \phi_\theta \) and \( k_\phi \). This treatment corresponds to an extension of Takata et al. (2004, hereafter TSH04; 2006, hereafter TSHC06), who solved the basic equations in the two-dimensional configuration space (see table 1), into the four-dimensional phase space (2-D configuration and 2-D momentum spaces). In § 4.1, we derive the Poisson equation that is applicable to arbitrary axisymmetric space-time and to arbitrary magnetic field configurations. We then present the Boltzmann equations for \( e^{\pm} \)’s and \( \gamma \)-rays, and apply the scheme to the Crab pulsar.
4.1 Poisson Equation

Around a rotating neutron star with mass $M$, the background space-time geometry is given by (Lense & Thirring 1918)

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt d\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2,$$  \hspace{1cm} (51)

where

$$g_{tt} \equiv \left(1 - \frac{r_g}{r}\right)c^2, \quad g_{t\phi} \equiv ac \frac{r_g}{r} \sin^2 \theta, \quad g_{rr} \equiv -\left(1 - \frac{r_g}{r}\right)^{-1}, \quad g_{\theta\theta} \equiv -r^2, \quad g_{\phi\phi} \equiv -r^2 \sin^2 \theta;$$ \hspace{1cm} (52, 53)

$r_g = 2GM/c^2$ indicates the Schwarzschild radius, and $a \equiv I\Omega/(Mc)$ parameterizes the stellar angular momentum. At radial coordinate $r$, the inertial frame is dragged at angular frequency $\omega \equiv -g_{t\phi}/g_{\phi\phi} = 0.15I_{45}r_6^{-3}$, where $I_{45} \equiv 10^{45}$ erg cm$^2$, and $r_6 \equiv r_s/10$ km.

Let us consider the Gauss’s law,

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \delta_\mu^\nu \left[ \frac{\sqrt{-g}}{\rho_w^2} g^{\mu\nu} (-g_{\phi\phi} F_{t\nu} + g_{t\phi} F_{\phi\nu}) \right] = \frac{4\pi}{c^2} \rho,$$ \hspace{1cm} (54)

where $\nabla$ denotes the covariant derivative, the Greek indices run over $t, r, \theta, \phi$; $\sqrt{-g} = \sqrt{g_{tt}g_{\theta\theta}g_{\phi\phi}} = cr^2 \sin \theta$ and $\rho_w^2 \equiv g_{t\phi}g_{\phi\phi}$, $\rho$ the real charge density. The electromagnetic fields observed by a distant static observer are given by (Camenzind 1986a, b) $E_r = F_{rt}, \quad E_\theta = F_{t\theta}, \quad E_\phi = F_{t\phi}, \quad B^r = (g_{tt} + g_{t\phi}\Omega)F_{\phi\nu}/\sqrt{-g}, \quad B^\theta = (g_{tt} + g_{t\phi}\Omega)F_{t\nu}/\sqrt{-g}, \quad B_\phi = -\rho_w^2 F_{t\phi}/\sqrt{-g}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu$ denotes the vector potential.

Assuming that the electromagnetic fields are unchanged in the corotating frame, we can introduce the non-corotational potential $\Psi$ such that

$$F_{\mu\nu} + \Omega F_{\mu\phi} = -\partial_\nu \Psi(r, \theta, \phi - \Omega t),$$ \hspace{1cm} (55)

where $\mu = t, r, \theta, \phi$. If $F_{At} + \Omega F_{A\phi} = 0 \ (A = r, \theta)$ holds, $\Omega$ is conserved along the field line. On the neutron-star surface, we impose $F_{t\theta} + \Omega F_{t\phi} = 0 \ (A = r, \theta)$ to find that the surface is equipotential, that is, $\partial_\theta \Psi = \partial_\phi \Psi + \Omega \partial_\phi \Psi = 0$ holds. However, in a particle acceleration region, $F_{At} + \Omega F_{A\phi}$ deviates from 0 and the magnetic field does not rigidly rotate. The deviation is expressed in terms of $\Psi$, which gives the strength of the acceleration electric field that is measured by a distant static observer as

$$E_\parallel \equiv \frac{B}{B} \cdot \mathbf{E} = \frac{B^i}{B} (F_{it} + \Omega F_{i\phi}) = \frac{B}{B} \cdot (-\nabla \Psi),$$ \hspace{1cm} (56)

where the Latin index $i$ runs over spatial coordinates $r, \theta, \phi$.

Substituting equation (55) into (54), we obtain the Poisson equation for the non-corotational potential,

$$-\frac{c^2}{\sqrt{-g}} \partial_\mu \left( \frac{1}{\rho_w^2} g^{\mu\nu} g_{\phi\phi} \partial_\nu \Psi \right) = 4\pi (\rho - \rho_{GJ}),$$ \hspace{1cm} (57)

where the general relativistic Goldreich-Julian charge density is defined as

$$\rho_{GJ} \equiv \frac{c^2}{4\pi \sqrt{-g}} \partial_\mu \left[ \frac{\sqrt{-g}}{\rho_w^2} g^{\mu\nu} (\Omega - \omega) F_{\phi\nu} \right].$$ \hspace{1cm} (58)
If $p$ deviates from $p_{GJ}$ in any region, $E_{||}$ is exerted along $B$. In the limit $r \gg r_G$, equation (58) reduces to the ordinary, special-relativistic expression (Goldreich and Julian 1969; Mestel 1971),

$$\rho_{GJ} \equiv -\frac{\Omega \cdot B}{2\pi c} + \frac{(\Omega \times r) \cdot (\nabla \times B)}{4\pi c}. \quad (59)$$

Instead of $(r, \theta, \phi)$, it is convenient to adopt the magnetic coordinates $(x, \theta, \phi_x)$, which was introduced in § 2. With this coordinate system, we obtain the following form of Poisson Eq., which can be applied to arbitrary magnetic field configurations (Paper XI):

$$-\frac{e^2}{\rho_w^2} g_{\phi \phi} \left( g_{xx} \frac{\partial^2}{\partial x^2} + g_{y0} \frac{\partial^2}{\partial y \partial \theta} + g_{z\phi} \frac{\partial^2}{\partial z \partial \phi_x} + 2g_{xy} \frac{\partial}{\partial y} + 2g_{xz} \frac{\partial}{\partial z} + 2g_{z\phi} \frac{\partial}{\partial \phi_x} \right) \Psi + \left(A^i \partial_i + A^\theta \partial_\theta + A^\phi \partial_\phi \right) \Psi = 4\pi (\rho - \rho_{GJ}). \quad (60)$$

$$g_{ff} = g_{ff} \frac{\partial^2}{\partial x^2} + g_{ff} \frac{\partial^2}{\partial y \partial \theta} + g_{ff} \frac{\partial^2}{\partial z \partial \phi_x} - \frac{k_0}{\rho_w^2} \frac{\partial}{\partial \phi} \frac{\partial \theta}{\partial \phi_x}, \quad (61)$$

$$A^f \equiv \frac{e^2}{\sqrt{-g}} \left( \frac{\partial}{\partial r} \left[ \frac{g_{\phi \phi}}{g_{\phi \phi}} \sqrt{-g} \frac{\partial \chi^f}{\partial r} \right] + \frac{\partial}{\partial \theta} \left[ \frac{g_{\phi \phi}}{g_{\phi \phi}} \sqrt{-g} \frac{\partial \chi^f}{\partial \theta} \right] \right) - \frac{c^2 g_{\phi \phi}}{\rho_w^2} k_0 \frac{\partial^2 \chi^f}{\partial \phi^2}, \quad (62)$$

where $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^f = s$, $x^f = \theta$, and $x^j = \phi_x$. The light surface, a generalization of the light cylinder, is obtained by setting $k_0 \equiv k_0 + 2g_{\phi \phi} \Omega + g_{\phi \phi} \Omega^2$ to be zero (e.g., Znajek 1977; Takahashi et al. 1990). Equation (56) gives $E_{||} = -\frac{\partial \Psi}{\partial s} \theta_x \phi$. Magnetic field expansion effect is contained in $g_{\theta \theta}$, $g_{\phi \phi}$, $g_{\phi \phi}$. Magnetic field expansion effect is contained in $g_{\theta \theta}$, $g_{\phi \phi}$, $g_{\phi \phi}$.

### 4.2 Particle Boltzmann Equations

In the same way as we derived equation (12), we can reduce the particle Boltzmann equations along each magnetic field line (i.e., for a constant $\theta_x$) as

$$c \cos \chi \frac{\partial n_{\pm}}{\partial s} + \frac{dp}{dt} \frac{\partial n_{\pm}}{\partial p} + \frac{d\chi}{dt} \frac{\partial n_{\pm}}{\partial \chi} = S_{\pm}, \quad (63)$$

where the upper and lower signs correspond to the positrons (with charge $q = +e$) and electrons ($q = -e$), respectively; $p \equiv |p|$ and

$$\frac{dp}{dt} \equiv qE_{||} \cos \chi - \frac{P_{SC}}{c} \quad (64)$$

$$\frac{d\chi}{dt} \equiv -\frac{qE_{||}}{p} \sin \chi + c \frac{\partial (\ln B^{1/2})}{\partial s} \sin \chi, \quad (65)$$

$$\frac{ds}{dt} = c \cos \chi. \quad (66)$$

For outward- (or inward-) migrating particles, $\cos \chi > 0$ (or $\cos \chi < 0$). Since we consider relativistic particles, we obtain $\Gamma = p/(m_e c)$. The second term in the right-hand side of equation (65) shows that the particle’s pitch angle evolves due to the the variation of $B$ (e.g., § 12.6 of Jackson 1962). For example, without $E_{||}$, inward-migrating particles would
be reflected by the magnetic mirrors. Using \( n_\pm \), we can express the leptonic charge density \( \rho_e \) as

\[
\rho_e = \frac{\Omega B}{2\pi c} \int [n_+(s, \theta, \phi, \Gamma, \chi) - n_-(s, \theta, \phi, \Gamma, \chi)] d\Gamma d\chi.
\]  

(67)

The radiation-reaction force due to synchro-curvature radiation is given by (Cheng & Zhang 1996; ZC97),

\[
P_{\text{SC}} = \frac{e^2 \Gamma^4 Q_2}{12 r_c} \left( 1 + \frac{7}{r_c^2 Q_2^2} \right),
\]  

(68)

where

\[
r_c \equiv \frac{c^2}{(n_B + \rho_e) (c \cos \chi / \rho_e)^2 + n_B \omega_B^2},
\]  

(69)

\[
Q_2^2 \equiv \frac{1}{r_B} \left( \frac{r_B^2 + \rho_e r_B - 3 \rho_e^3}{\rho_e^4} \cos^4 \chi + \frac{3}{\rho_e} \cos^2 \chi + \frac{1}{r_B} \sin^4 \chi \right),
\]  

(70)

\[
r_B = \frac{\Gamma m_e c^2 \sin \chi}{eB}, \quad \omega_B = \frac{eB}{\Gamma m_e c}
\]  

(71)

and \( \rho_e \) is the curvature radius of the magnetic field line. In the limit of \( \chi \to 0 \) (or \( \rho_e \to \infty \)), equation (68) becomes the expression of pure-curvature (or pure synchrotron) emission.

We briefly comment on the inclusion of the radiation-reaction force, \( P_{\text{SC}}/c \), in equation (64). Except for the vicinity of the star, the magnetic field is much less than the critical value \( (B_{cr} \equiv 4.41 \times 10^{13} \, \text{G}) \) so that quantum effects can be neglected in synchrotron radiation. Thus, we regard the radiation-reaction force, which is continuous, as an external force acting on a particle. Near the star, if \( \Gamma (B/B_{cr}) \sin \chi > 0.1 \) holds, the energy loss rate decreases from the classical formula (Erber et al. 1966). If \( \Gamma (B/B_{cr}) \sin \chi > 1 \) holds very close to the star, the particle motion perpendicular to the field is quantized and the emission is described by the transitions between Landau states; thus, equation (64) and (68) breaks down. In this case, we artificially put \( \chi = 10^{-20} \), which guarantees pure-curvature radiation after the particles have fallen onto the ground-state Landau level, avoiding to discuss the detailed quantum effects in the strong-\( B \) region. This treatment will not affect the main conclusions of this chapter, because the gap electrodynamics is governed by the pair creation taking place not very close to the star.

Let us describe some details of the collision terms \( S_\pm \), which are given by equation (11). If we multiply \( d\Gamma \) on both sides of equation (11), the first (or the second) term in the right-hand side represents the rate of particles disappearing from (or appearing into) the energy interval \( m_e c^2 \Gamma \) and \( m_e c^2 (\Gamma + d\Gamma) \) due to inverse-Compton (IC) scatterings; the third term does the rate of two-photon and one-photon pair creation processes.

The IC redistribution function \( \eta_{\text{IC}}^c(E_\gamma; \Gamma, \mu_e) \) represents the probability that a particle with Lorentz factor \( \Gamma \) upscatters photons into energies between \( E_\gamma \) and \( E_\gamma + dE_\gamma \) per unit time when the collision angle is \( \cos^{-1} \mu_e \). On the other hand, \( \eta_{\text{IC}}^c(\Gamma_i, \Gamma_f, \mu_e) \) describes the probability that a particle changes Lorentz factor from \( \Gamma_i \) to \( \Gamma \) in a scattering. Thus, energy conservation gives

\[
\eta_{\text{IC}}^c(\Gamma_i, \Gamma_f, \mu_e) = \eta_{\text{IC}}^c(\Gamma_i - \Gamma_f, m_e c^2, \Gamma_i, \mu_e).
\]  

(72)
The quantity $\eta_{IC}^\gamma$ is defined by the soft photon flux $dF_s/dE_s$ and the Klein-Nishina cross section $\sigma_{KN}$ as follows (Paper X):

$$\eta_{IC}^\gamma(E_{\gamma}, \Gamma, \mu_c) = (1 - \beta \mu_c) \times \int_{E_{\min}}^{E_{\max}} \frac{dF_s}{dE_s} \int_{b_{i-1}}^{b_i} \frac{dE_{\gamma}'}{dE_{\gamma}} \int_{-1}^{1} \frac{d\sigma_{KN}(E_{\gamma}', \Gamma, \mu_c)}{d\Omega_{\gamma}} \frac{d\Omega_{\gamma}}{dE_{\gamma}} \frac{dE_{\gamma}}{dE_{\gamma}'}$$  \hspace{1cm} (73)

where $\beta = \sqrt{1 - 1/\gamma^2}$ is virtually unity, $\Omega_{\gamma}$ the solid angle of upscattered photon, the prime denotes the quantities in the electron (or positron) rest frame, and $E_{\gamma} = (b_{i-1} + b_i)/2$. In the rest frame of a particle, a scattering always takes place well above the resonance energy. Thus, the Klein-Nishina cross section can be applied to the present problem. The soft photon flux per unit photon energy $E_s$ [s$^{-1}$ cm$^{-2}$] is written as $dF_s/dE_s$ and is given by the surface blackbody emission with redshift corrections at each distance from the star.

The differential pair-creation redistribution function is given by

$$\frac{\partial \eta_{p\beta}}{\partial \Gamma}(E_{\gamma}, \Gamma, \mu_c) = (1 - \mu_c) \int_{E_{\min}}^{E_{\max}} \frac{dF_s}{dE_s} \frac{d\sigma_p(E_{\gamma}, \Gamma, \mu_c)}{d\Gamma}, \hspace{1cm} (74)$$

where the pair-creation threshold energy is defined by equation (32), and the differential cross section is given by

$$\frac{d\sigma_p}{d\Gamma} = \frac{3}{8} \frac{1 - \beta_{CM}^2}{E_{\gamma}} \times \left[ \frac{1 + \beta_{CM}^2 (2 - \mu_{CM}^2)}{1 - \beta_{CM}^2 \mu_{CM}^2} - \frac{2 \beta_{CM}^4 (1 - \mu_{CM}^2)^2}{(1 - \beta_{CM}^2 \mu_{CM}^2)^2} \right]; \hspace{1cm} (75)$$

$\sigma_T$ refers to the Thomson cross section and the center-of-mass quantities are defined as

$$\mu_{CM} \equiv \pm \frac{2 \Gamma m_e c^2 - E_{\gamma}}{\beta_{CM} E_{\gamma}}, \hspace{0.5cm} \beta_{CM}^2 \equiv 1 - \frac{2 (m_e c^2)^2}{(1 - \mu_c) E_s E_{\gamma}}. \hspace{1cm} (76)$$

Since a convenient form of $\partial \eta_{p\beta}/\partial \Gamma$ is not given in the literature, we simply assume that all the particles are created at the energy $\Gamma m_e c^2 = E_{\gamma}/2$ for magnetic pair creation. This treatment does not affect the conclusions in this chapter.

Let us briefly mention the electric current per magnetic flux tube. With projected velocities, $c \cos \chi$, along the field lines, electric current density in units of $\Omega B/(2\pi)$ is given by

$$j_{\text{gap}}(s, \theta_s) = j_e(s, \theta_s) + j_{\text{ion}}(\theta_s), \hspace{1cm} (77)$$

where

$$j_e \equiv \iint (n_- + n_+) \cos \chi dp d\chi; \hspace{1cm} (78)$$

$j_{\text{ion}}$ denotes the current density carried by the ions emitted from the stellar surface. Since $dp/dt$ and $d\chi/dt$ in equation (63) depend on momentum variables $p$ and $\chi$, $j_e$ and hence $j_{\text{gap}}$ does not conserve along the field line in an exact sense. Nevertheless, $j_{\text{gap}}$ is virtually kept constant for $s$, because the returning particles, of which pitch angles satisfy $|\cos \chi| \ll 1$, occupy only a small population at each point.
4.3 Gamma-ray Boltzmann Equations

Imposing the stationary condition (10), or equivalently, assuming that \( g \) depends on \( \varphi \) and \( t \) as \( g = g(r, \theta, \varphi - \Omega t, k) \), we obtain

\[
\left( \frac{c k^\varphi}{|k|} - \Omega \right) \frac{\partial g}{\partial \varphi} + \frac{c k^\theta}{|k|} \frac{\partial g}{\partial \theta} + \frac{c k^0}{|k|} \frac{\partial g}{\partial t} = S_g(r, \theta, \varphi, c|k|, k^\theta, k^\varphi),
\]

where \( \varphi = \varphi - \Omega t \). To compute \( k^i \), we have to solve the photon propagation in the curved spacetime. Since the wavelength is much shorter than the typical system scales, geometrical optics gives the evolution of momentum and position of a photon by the Hamilton-Jacobi equations,

\[
\frac{dk_{\lambda}}{d\lambda} = -\frac{\partial k_\theta}{\partial \varphi}, \quad \frac{dk_\theta}{d\lambda} = -\frac{\partial k_t}{\partial \varphi}, \quad \frac{dr}{d\lambda} = \frac{\partial k_t}{\partial \theta}, \quad \frac{d\theta}{d\lambda} = \frac{\partial k_t}{\partial \varphi},
\]

where the parameter \( \lambda \) is defined so that \( c d\lambda \) represents the distance (i.e., line element) along the ray path. The photon energy at infinity \( k_t \) and the azimuthal wave number \(-k_\varphi\) are conserved along the photon path in a stationary and axisymmetric spacetime (e.g., in the spacetime described by eqs. [51]–[53]). Hamiltonian \( k_t \) can be expressed in terms of \( k_r, k_\theta, k_\varphi, r, \theta, \varphi \) from the dispersion relation \( k^\mu k_\mu = 0 \), which is a quadratic equation of \( k_\mu \) \((\mu = t, r, \theta, \varphi)\). Thus, we have to solve the set of four ordinary differential equations (80) and (81) for the four quantities, \( k_r, k_\theta, k_\varphi, r, \) and \( \theta \) along the ray. Initial conditions at the emitting point are given by \( k^i/|k| = \pm B^i/|B| \), where \( i = r, \theta, \varphi \); the upper (or lower) sign is chosen for the \( \gamma \)-rays emitted by an outward- (or inward-) migrating particle. When a photon is emitted with energy \( E_{\text{local}} \) by the particle of which angular velocity is \( \varphi \), it is related with \( k_t \) and \(-k_\varphi\) by the redshift relation, \( E_{\text{local}} = (dt/d\tau)(k_t + k_\varphi \varphi) \), where \( dt/d\tau \) is solved from \( (dt/d\tau)^2 (g_{tt} + 2g_{t\varphi} \varphi + g_{\varphi\varphi} \varphi^2) = 1 \). To express the energy dependence of \( g \), we regard \( g \) as a function of \( k_t = E_\gamma \) (i.e., observed photon energy).

In this chapter, in accordance with the two-dimensional analysis of equations (60) and (63), we neglect \( \varphi \) dependence of \( g \), by ignoring the first term in the left-hand side of equation (79). In addition, we neglect the aberration of photons and simply assume that the \( \gamma \)-rays do not have angular momenta and put \( k_\varphi = 0 \). The aberration effects are important when we discuss how the outward-directed \( \gamma \)-rays will be observed. However, they can be correctly taken into account only when we compute the propagation of emitted photons in the three-dimensional magnetosphere. Moreover, they are not essential when we investigate the electrodynamics, because the pair creation is governed by the specific intensity of inward-directed \( \gamma \)-rays, which are mainly emitted in a relatively inner region of the magnetosphere. Thus, it seems reasonable to adopt \( k_\varphi = 0 \) when we investigate the two-dimensional gap electrodynamics.

We linearly divide the longitudinal distance into 400 grids from \( s = 0 \) (i.e., stellar surface) to \( s = 1.4\delta s_{p, c} \), and the meridional coordinate into 16 field lines from \( \theta_s = \theta_{s_{\text{max}}} \) (i.e., the last-open field line) to \( \theta_s = \theta_{s_{\text{min}}} \) (i.e., gap upper boundary), and consider only \( \varphi_s = 0 \) plane (i.e., the field lines threading the stellar surface on the plane formed by the rotation and magnetic axes). To solve the particle Boltzmann equations (63), we adopt the Cubic Interpolated Propagation (CIP) scheme with the fractional step technique to shift the profile.
of the distribution functions $n_{\pm}$ in the direction of the velocity vector in the two-dimensional momentum space. To solve the $\gamma$-ray Boltzmann equation (79), on the other hand, we do not have to compute the advection of $g$ in the momentum space, because only the spatial derivative terms remain after integrating over $\gamma$-ray energy bins, which are logarithmically divided from $\beta_1 = 0.511$ MeV to $\beta_{29} = 28.7$ TeV into 29 bins. The $\gamma$-ray propagation directions, $k^\theta/k'$, are divided linearly into 180 bins every $\Delta \theta = 2$ degrees. Since the specific intensity in $i$th energy bin at height $\theta_s = \theta_s^{max} - (k/16)(\theta_s^{max} - \theta_s^{min})$, is given by

$$ g_{i,k}(s) = \frac{c}{\Delta \theta \Delta \phi} \int_{b_i}^{b_{i+1}} g^{(k)}(s, \theta_s, E, (k^\theta/k')) dE, \quad (82) $$

the observed $\gamma$-ray energy flux at distance $d$ is calculated as

$$ F_{i,l} = \frac{\Delta \psi \sum_k \Delta \varphi g_{i,k,l}}{d^2}, \quad (83) $$

where $\Delta \psi$ denotes the azimuthal dimension of the gap at longitudinal distance $s$ (=$ \sigma_{1c}$ in this chapter), $\Delta \varphi$ the meridional thickness between two field lines with $\theta_s = \theta_{s,k}$ and $\theta_{s,k+1}$, and $i = 1, 2, 3, \ldots, 28$, $k = 1, 2, 3, \ldots, 15$, $l = 1, 2, 3, \ldots, 180$. To compute the phase-averaged spectrum, we set the azimuthal width of the $\gamma$-ray propagation direction, $\Delta \phi$, to be $\pi$ radian.

Equation (15) describes the $\gamma$-ray absorption and creation rate within the gap. However, to compute observable fluxes, we also have to consider the synchrotron emission by the secondary, tertiary, and higher-generation pairs that are created outside of the gap. If an electron or positron is created with energy $\Gamma_0 mc^2$ and pitch angle $\chi_0$, it radiates the following number of $\gamma$-rays (in units of $\Omega B_s/2\pi c^2$) in energies between $b_{i-1}$ and $b_i$:

$$ \frac{d g_i}{dE} = \frac{2\pi c e}{\Omega B_s} \int_0^{\infty} dt \int_{b_{i-1}}^{b_i} \frac{dW}{E \Gamma dE}, \quad (84) $$

where

$$ dW = \frac{\sqrt{3} e^2 B \sin \chi_0}{h m e^2} F \left( \frac{E}{E_c} \right), \quad F(x) = x \int_x^{\infty} K_{5/3}(\xi) d\xi, \quad (85) $$

$$ \frac{d \Gamma}{dt} = -\frac{2}{3} \frac{e^4 B^2 \sin^2 \chi_0}{m_c^3 c^3} \Gamma^2, \quad (86) $$

$K_{5/3}$ is the modified Bessel function of $5/3$ order, and $E_c \equiv (3h/4\pi)(eB^2 \sin \chi_i)/(m_c)$ is the synchrotron critical energy at Lorentz factor $\Gamma$. Substituting equations (85) and (86) into (84), we obtain

$$ \frac{d g_i}{dE} = \frac{2\pi c e}{\Omega B_s} \frac{3\sqrt{3} m^2 c^3}{2heb \sin \chi_0} \int_{\Gamma_0}^{\Gamma_0/d\Gamma} \int_{b_{i-1}/E_c}^{b_i/E_c} \frac{d\Gamma}{\Gamma^2} \int_0^{\infty} K_{5/3}(\xi) d\xi. \quad (87) $$

Note that we assume that particle pitch angle is fixed at $\chi = \chi_0$, because ultra-relativistic particles emit radiation mostly in the instantaneous velocity direction, preventing pitch-angle evolution. Once particles lose sufficient energies, they preferentially lose perpendicular momentum; nevertheless, such less-energetic particles hardly emit synchrotron photons above MeV energies. On these grounds, to incorporate the synchrotron radiation of higher-generation pairs created outside of the gap, we add $\int \frac{dn}{d\Gamma_0} (d\Gamma_0) (d g_i/dE) d\Gamma_0$ to compute the emission of $\gamma$-rays in the energy interval $[b_{i-1}, b_i]$ in the right-hand side of equation (14), where $dn/d\Gamma_0$ denotes the particles created between position $s$ and $s + ds$ in Lorentz factor interval $[\Gamma_0, \Gamma_0 + d\Gamma_0]$. 
4.4 Boundary Conditions

In order to solve the set of partial differential equations (60), (63), and (79) for $\Psi$, $n_{\pm}$, and $g$, we must impose appropriate boundary conditions. We assume that the gap lower boundary (Fig. 2), $\theta_s = \theta_s^{\text{max}}$, coincides with the last open field line, which is defined by the condition that $\sin \theta \sqrt{-g r r B'} + \cos \theta \sqrt{-g a a B} = 0$ is satisfied at the light cylinder on the surface $\varphi_s = 0$. Moreover, we assume that the upper boundary coincides with a specific magnetic field line and parameterize this field line with $\theta_s = \theta_s^{\text{min}}$. In general, $\theta_s^{\text{min}}$ is a function of $\varphi_s$; however, we consider only $\varphi_s = 0$ in this chapter. Determining the upper boundary from physical consideration is a subtle issue, which is beyond the scope of this chapter. Therefore, we treat $\theta_s^{\text{min}}$ as a free parameter. We measure the trans-field thickness of the gap with

$$h_\text{m} = \frac{\theta_s^{\text{max}} - \theta_s^{\text{min}}}{\theta_s^{\text{max}}}. \quad (88)$$

If $h_\text{m} = 1.0$, it means that the gap exists along all the open field lines. On the other hand, if $h_\text{m} \ll 1$, the gap becomes transversely thin and $\theta_s$ derivatives dominate in equation (60). To describe the trans-field structure, we introduce the fractional height as

$$h = \frac{\theta_s^{\text{max}} - \theta_s}{\theta_s^{\text{max}}}. \quad (89)$$

Thus, the lower and upper boundaries are given by $h = 0$ and $h = h_\text{m}$, respectively.

The inner boundary is assumed to be located at the stellar surface. For the outer boundary, we solve the Poisson equation to a large enough distance, $s = 1.4 \sigma_{\text{LC}}$, which is located outside of the light cylinder. This mathematical outer boundary is introduced only for convenience in order that the $E_\parallel$ distribution inside of the light cylinder may not be influenced by the artificially chosen outer boundary position when we solve the Poisson equation. Since the structure of the outer-most part of the magnetosphere is highly unknown, we artificially set $E_\parallel = 0$ if the distance from the rotation axis, $\sigma$, becomes greater than $0.90 \sigma_{\text{LC}}$. Under this artificially suppressed $E_\parallel$ distribution in $\sigma > 0.90 \sigma_{\text{LC}}$, we solve the Boltzmann equations for outward-propagating particles and $\gamma$-rays in $0 < s < 1.4 \sigma_{\text{LC}}$. For inward-propagating particles and $\gamma$-rays, we solve only in $0 < s < 0.90 \sigma_{\text{LC}}$. The position of the mathematical outer boundary ($1.4 \sigma_{\text{LC}}$ in this case), little affects the results by virtue of the artificial boundary condition, $E_\parallel = 0$ for $\sigma > 0.9 \sigma_{\text{LC}}$. On the other hand, the artificial outer boundary condition, $E_\parallel = 0$ for $\sigma > 0.9 \sigma_{\text{LC}}$, affects the calculation of outward-directed $\gamma$-rays to some degree; nevertheless, it little affects the electrodynamics in the inner part of the gap ($s < 0.5 \sigma_{\text{LC}}$), which is governed by the absorption of inward-directed $\gamma$-rays.

First, to solve the elliptic-type equation (60), we impose $\Psi = 0$ on the lower, upper, and inner boundaries. At the mathematical outer boundary ($s = 1.4 \sigma_{\text{LC}}$), we impose $\partial \Psi / \partial s = 0$. Generally speaking, the solved $E_\parallel = -(\partial \Psi / \partial s)_{s=0}$ under these boundary conditions does not vanish at the stellar surface. Let us consider how to cancel this remaining electric field.

For a super-GJ current density in the sense that $\rho_e - \rho_{\text{GJ}} < 0$ holds at the stellar surface, equation (60) gives a positive electric field near the star. In this case, we assume that ions are emitted from the stellar surface so that the additional positive charge in the thin non-relativistic region may bring $E_\parallel$ to zero (for the possibility of free ejection of ions due to a low work function, see Jones 1985, Neuhauser et al. 1986, 1987). The column density in
the non-relativistic region becomes (SAF78)
\[ \Sigma_{NR} = \frac{1}{2\pi} \sqrt{\frac{c\Omega B_s}{q/m}} j_{ion}, \]
(90)
where \( q/m \) represents the charge-to-mass ratio of the ions and \( j_{ion} \) the ionic current density in units of \( \Omega B_s/(2\pi) \). Equating \( 4\pi \Sigma_{NR} \) to \( -(\partial \Psi / \partial s)_{s=0} \) calculated from relativistic positrons, electrons and ions, we obtain the ion injection rate \( j_{ion} \) that cancels \( E_\parallel \) at the stellar surface.

For a sub-GJ current density in the sense that \( \rho_e - \rho_{GJ} > 0 \) holds at the stellar surface, \( \Psi \) increases outwards near the star to peak around \( s = 0.02 \sigma_{LC} \sim 0.1 \sigma_{LC} \), depending on \( \alpha_i \) and \( \rho_e(s=0) \), then decrease to become negative in the outer magnetosphere. That is, \( -(\partial \Psi / \partial s)_{s=0} < 0 \) holds in the inner region of the gap. In this case, we assume that electrons are emitted from the stellar surface and fill out the region where \( \Psi > 0 \); thus, we artificially put \( \Psi = 0 \) if \( \Psi > 0 \) appears. Even though a non-vanishing, positive \( E_\parallel \) is remained at the inner boundary, which is located away from the stellar surface, we neglect such details. This is because the gap with a sub-GJ current density is found to be inactive and hence less important, as will be demonstrated in the next section.

Secondly, to solve the hyperbolic-type equations (63) and (79), we assume that neither positrons nor \( \gamma \)-rays are injected across the inner boundary; thus, we impose
\[ n_+(s_{in}, \theta_s, \Gamma, \chi) = 0, \quad g(s_{in}, \theta_s, E_\gamma, \theta_\gamma) = 0 \]
(91)
for arbitrary \( \theta_s, \Gamma, 0 < \chi < \pi/2, E_\gamma \), and \( \cos(\theta_\gamma - \theta_B) > 0 \), where \( \theta_B \) designates the outward magnetic field direction. In the same manner, at the outer boundary, we impose
\[ n_-(s_{out}, \theta_s, \Gamma, \chi) = 0, \quad g(s_{out}, \theta_s, E_\gamma, \theta_\gamma) = 0 \]
(92)
for arbitrary \( \theta_s, \Gamma, \pi/2 < \chi < \pi, E_\gamma \), and \( \cos(\theta_\gamma - \theta_B) < 0 \).

4.5 Application to the Crab Pulsar

We next apply the scheme to the Crab pulsar, adopting four free parameters, \( \alpha_i, \mu, kT_s \), and \( h_{in} \). Other quantities such as gap geometry on the poloidal plane, exerted \( E_\parallel \) and potential drop, particle density and energy distribution, the \( \gamma \)-ray flux and spectrum, as well as the created pairs outside of the gap, are uniquely determined if we specify these four parameters.

The Crab pulsar has been studied from radio, optical, X-ray to \( \gamma \)-ray wavelength since its discovery (Staelin & Reifenstein 1968; Comella et al. 1969). Its period and period derivative are \( P = 33.0 \) ms and \( \dot{P} = 4.20 \times 10^{-13} \) s s\(^{-1}\). Using magnetic dipole radiation formula for an orthogonal rotator, the observed spin down luminosity \( 4.46 \times 10^{38} \) ergs s\(^{-1}\) gives \( \mu = 3.80 \times 10^{30} (d/2 \text{ kpc})^2 \). From soft X-ray observations, 180 eV is obtained (Tennant et al. 2001) as the upper limit of the cooling neutron-star blackbody temperature, \( kT \).

4.5.1 Sub-GJ Solution

Adopting \( kT = 100 \) eV, \( \mu = 4.0 \times 10^{30} \) G cm\(^3\), and the magnetic inclination \( \alpha_i = 70^\circ \), which is more or less close to the value (65\(^\circ\)) suggested by a three-dimensional analysis in the
traditional outer gap model, we obtain a nearly vacuum solution (fig. 8) for a geometrically thin case $h_{m} = 0.047$. In the left panel, we present $E_{\parallel}(s,h)$ at five discrete heights. The solutions become similar to the vacuum one obtained in CHR86a,b. For one thing, the inner boundary is located slightly inside of the null surface. What is more, $E_{\parallel}$ maximizes at the central height, $h = h_{m}/2$, and remains roughly constant in the entire region of the gap. The solved $E_{\parallel}$ distributes almost symmetrically with respect to the central height; for example, the dashed and dash-dotted curves nearly overlap each other. The gap has no outer termination within the light cylinder. Since the inner boundary coincides with the place where $\Psi$ vanishes, the region between the star and the inner boundary has $\Psi > 0$. Therefore, negative charges pulled from the stellar surface with $E_{\parallel} = -\partial_{s}\Psi < 0$ populate only inside of the inner boundary, in which $\Psi < 0$ holds. Similar solutions are obtained for a thinner gap, $h_{m} < 0.047$, even though there appears a small $E_{\parallel}$ peak near the null surface, which is less important.

CHR86a,b first considered this kind of vacuum solutions and suggested two outer gaps can be formed for each magnetic pole Considering a fan beam (instead of a pencil beam in the inner-gap model or a funnel beam in the inner-slot-gap model) as the emission morphology, they discussed the formation of cusped photon peak. Extending this morphological emission model, Stanford group (RY95) discussed the observed properties of individual $\gamma$-ray pulsars, their radio to $\gamma$-ray pulse offsets, and the radio- vs. $\gamma$-ray detection probabilities. Assuming $E_{\parallel} \propto s^{-1}$ and a power-law energy distribution accelerated $e^{\pm}$s, R96 estimated

Figure 8: Traditional outer-gap solution obtained for the Crab pulsar with $\alpha_{i} = 70^\circ$ and $h_{m} = 0.047$. Left: The field-aligned electric field at discrete heights $h$ ranging from $2h_{m}/16$, $5h_{m}/16$, $8h_{m}/16$, $11h_{m}/16$, $14h_{m}/16$, with dashed, dotted, solid, dash-dot-dot-dot, and dash-dotted curves, respectively. The abscissa indicates the distance along the field line from the star in the unit of the light-cylinder radius. The null surface position at the height $h = h_{m}/2$ is indicated by the down arrow. Right: Calculated phase-averaged spectra of the pulsed, outward-directed $\gamma$-rays. The flux is averaged over the meridional emission angles (see Paper XI for details).
the evolution of high-energy flux efficiencies and beaming fractions to discuss the detection statistics. Another group in Hong Kong (ZC97) examined the minimum trans-field thickness, $h_m$, of the gap, imposing that the $\gamma-\gamma$ pair creation criterion is met. They estimated the soft photon field emitted from the heated polar-cap surface by the bombardment of gap-accelerated charged particles, adopting essentially the same $E_\parallel$ solution as Fig. 8. Extending their work, CRZ00 developed a three-dimensional outer magnetospheric gap model to examine the double-peak light curves with strong inter-pulse emission, and estimated phase-resolved $\gamma$-ray spectra by assuming that the charged particles are accelerated to the Lorentz factors at which the curvature radiation-reaction force balances with the electrostatic acceleration.

The outer gap models of these two groups have been successful in explaining the observed light curves, particularly in reproducing the wide separation of the two peaks, without invoking a very small inclination angle (as in inner-gap models). However, if we solve Eq. (60) self-consistently with the particle and $\gamma$-ray Boltzmann equations (Paper XI), we find that the $\gamma$-ray flux obtained for $h_m < 0.047$ (i.e., traditional outer-gap models) is insufficient (right panel of Fig. 8). Thus, we have to consider a transversely thicker gap, which exerts a larger $E_\parallel$ because of the less-efficient screening due to the two zero-potential walls at $h = 0$ and $h_m$. If the created current increases due to the increased $h_m$, the gap inner boundary deviates the null surface and shifts inwards to touch the stellar surface at last (Paper X).

### 4.5.2 Super-GJ Solution

For $h_m > 0.047$, the created current density $j_c$ (see eq. [77]) becomes super Goldreich-Julian in the sense that $\rho < \rho_{GJ} < 0$ holds at the inner boundary (left panel of Fig. 9). The predicted $\gamma$-ray flux is much larger than the sub-GJ case of $h_m \leq 0.047$ (i.e., traditional outer-gap solution). Moreover, the copious pair creation leads to a substantial screening of $E_\parallel$ in the inner region, as the right panel shows. In this screening region, $\rho(s, h)$ distributes (Fig. 10) so that $E_\parallel$ may virtually vanish. Because of the ion emission from the stellar surface, the total charge density $\rho$ is given by $\rho = \rho_e + \rho_{ion}$, where $\rho_e$ denotes the sum of positronic and electronic charge densities, while $\rho_{ion}$ does the ionic one. We should notice here that even if $E_\parallel \approx 0$ occurs by the discharge of created pairs in most portions of the gap, the negative $\rho_e - \rho_{GJ}$ inevitably exerts a strong positive $E_\parallel$ at the surface (see Paper XI for details), thereby extracting ions from the surface for the solution with super-GJ current. On these grounds, we can regard this modern outer-gap solution as a mixture of the traditional inner-gap model, which extracts electrons from the surface with $E_\parallel < 0$, and the traditional outer-gap model, which exerts positive $E_\parallel$ because of the negativity of $\rho - \rho_{GJ}$.

There is space here only for a brief comments on the Lorentz-factor and pitch-angle dependence of particle distribution functions, $n_\pm$ (see Paper XI for details). First, mono-energetic approximation is not good for positrons, which migrate outwards. This is because pairs are created at various points in the gap and follow different characteristics (see solid curves in the right panel of Fig. 6), and because a small portions (a few percent) of positrons up-scatter the surface blackbody photons to lose energies. Positrons are mostly created with inward momenta initially and return by the positive $E_\parallel$ to lose most of their perpendicular momentum by synchrotron radiation. Thus, their radiation in the outer magnetosphere can
be safely approximated by the pure curvature formula. Secondly, electrons efficiently up-scatter surface photons (by more or less head-on collisions) to have a broad energy spectra. For example, at $s = 0.4\sigma_{\text{LC}}$, they broadly distribute in $10^4 < \Gamma < 10^7$ and $10^{-8} < \sin \chi < 10^{-3.5}$. It follows that the pure curvature formula completely breaks down for electrons, which migrate inwards in the inner magnetosphere ($B > 10^7$ G) and that we must adopt the synchro-curvature formula, instead of pure curvature one.

5 Discussion

In summary, we have quantitatively examined the stationary pair-creation cascade in an outer magnetosphere, by solving the set of Maxwell and Boltzmann equations. By one-dimensional analysis, it is revealed that the gap position shifts by the injection of charged particles across the boundary. For example, the gap is located near to the light cylinder (or the stellar surface) if the injection rate across the inner (or outer) boundary approaches the typical Goldreich-Julian value. This conclusion is, in fact, unchanged if the gap has a two-dimensional structure. It should be emphasized that the particle energy distribution is not represented by a power law, as assumed in some of previous outer-gap models. Moreover, applying the two-dimensional scheme to the Crab pulsar, we find that the solution represents the traditional outer-gap model if the created current density, $j_e$, is small compared to the Goldreich-Julian (GJ) value. However, in this case, the predicted $\gamma$-ray flux for the Crab pulsar is too small to explain the observed value. We find a new accelerator solution that has a super-GJ current density and extends from the stellar surface to the outer magnetosphere.
Figure 10: Modern outer-gap solution of the total (solid), created (dash-dotted), and Goldreich-Julian (dashed) charge densities in \( \Omega B(s, h)/(2\pi c) \) unit, for \( \alpha_i = 70^\circ \) and \( h_m = 0.060 \) at four transfield heights, \( h \). Because of an ion emission from the stellar surface, the total charge density deviates from the created one. From Paper XI.
This new solution possesses an acceleration electric field, $E_\parallel$, that is substantially screened in the inner part. However, the negative effective charge density, $\rho_{\text{eff}} \equiv \rho - \rho_{\text{GJ}}$, results in a non-vanishing, positive $E_\parallel$ in the inner-most region, which extracts ions from the surface. It is essential to examine the pitch-angle evolution of the created particles, because the inward-migrating particles emit $\gamma$-rays, which governs the gap electrodynamics through pair creation, via synchro-curvature process rather than pure-curvature one. The resultant spectral shape of the outward-directed $\gamma$-rays is consistent with the existing observations.

We consider the stability of such a gap in the next subsection. We then compare the new gap solution with an existing model.

5.1 Stability of the Gap

Let us discuss the electrodynamic stability of the gap, by considering whether an initial perturbation of some quantity tends to be canceled or not. In this chapter, we consider that the soft photon field is given and unchanged when gap quantities vary. Thus, let us first consider the case when the soft photon field is fixed. Imagine that the created pairs are decreased as an initial perturbation. It leads to an increase of the potential drop due to less efficient screening by the discharged pairs, and hence to an increase of particle energies. Then the particles emit synchro-curvature radiation efficiently, resulting in an increase of the created pairs, which tends to cancel the initial decrease of created pairs.

Let us next consider the case when the soft photon field also changes. Imagine again that the created pairs are decreased as an initial perturbation. It leads to an increase of particle energies in the same manner as discussed just above. The increased particle energies increase not only the number and density of synchro-curvature $\gamma$-rays, but also the surface blackbody emission from heated polar caps and the secondary magnetospheric X-rays. Even though neither the heated polar-cap emission nor the magnetospheric emission are taken into account as the soft photon field illuminating the gap in this chapter, they all work, in general, to increase the pair creation within the gap, which cancels the initial decrease of created pairs more strongly than the case of the fixed soft photon field.

Because of such negative feedback effects, solution exists in a wide parameter space. For example, the created current density is almost unchanged for a wide range of $h_m$ (e.g., compare the dash-dotted and dash-dot-dot-dotted curves in the left panel of fig. 9). On these grounds, although the perturbation equations are not solved under appropriate boundary conditions for the perturbed quantities, we conjecture that the particle accelerator is electrodynamically stable, irrespective whether the X-ray field illuminating the gap is thermal or non-thermal origin.

5.2 Comparison with Polar-slot gap model

It is worth comparing the present results with the polar-slot-gap model proposed by Muslimov and Harding (2003; 2004a,b, hereafter MH04a,b), who obtained a quite different solution (e.g., negative $E_\parallel$ in the gap) solving essentially the same equations under analogous boundary conditions for the same pulsar as in the present work. The only difference is the transfield thickness of the gap (i.e., $h_m$). Estimating the transfield thickness to be $\Delta l_{SG} \sim h_m r_s \sqrt{r/\Omega_{LC}}$, which is a few hundred times thinner than the present work, they extended the solution (near the polar cap surface) that was obtained by MT92 into the higher
altitudes (towards the light cylinder). Because of this very small $\Delta l_{SG}$, emitted $\gamma$-rays do not efficiently materialize within the gap; as a result, the created and returned positrons from the higher altitudes do not break down the original assumption of SCLF near the stellar surface.

To avoid the reversal of $E_\parallel$ in the gap (from negative near the star to positive in the outer magnetosphere), or equivalently, to avoid the reversal of the sign of the effective charge density, $\rho_{eff} = \rho - \rho_{GJ}$, along the field line, MH04a and MH04b assumed that $\rho_{eff}/B$ nearly vanishes and remains constant above a certain altitude, $s = s_c$, where $s_c$ is estimated to be within a few neutron star radii. Because of this assumption, $E_\parallel$ is suppressed at a very small value and the pair creation becomes negligible in the entire gap. In another word, the enhanced screening is caused not only by the proximity of two conducting boundaries, but also by the assumption of $\partial(\rho_{eff}/B)/\partial s = 0$ within the gap (see eq. [5]). To justify this $\rho/B$ distribution, MH04a and MH04b proposed an idea that $\rho$ should grow by the cross field motion of charges due to the toroidal forces, and that $\rho_{eff}/B$ is a small constant so that $c\rho_{eff}/B$ may not exceed the flux of the emitted charges from the star, which ensures the equipotentiality of the slot-gap boundaries (see § 2.2 of MH04a for details).

The cross-field motion becomes important if particles gain angular momenta as they migrate outwards to pick up energies which is a non-negligible fraction of the difference of the cross-field potential between the two conducting boundaries. Denoting the fraction as $\varepsilon$, we obtain $\Gamma m_e c^2 \phi \Omega (\sigma/c)^2 \equiv \varepsilon eB\Delta l_{SG}$ (Mestel 1985; eq. [12] of MH04a). If we substitute their estimate $\Delta l_{SG} \sim \gamma r_s/20$, we obtain $\varepsilon \sim 0.33(\phi/\Omega)\gamma B_6^{-1} r_s^{-3}(\sigma/\Omega_{LC})^2$, where $\gamma_7 = \gamma/10^7$, $B_6 = B_s/10^6$ G, and $r_s = r_s/10$ km; therefore, the cross-field motion becomes important in the outer magnetosphere within their (transversely very thin) slot-gap model.

As for the equipotentiality of the boundaries, it seems reasonable to suppose that $c|\rho_{eff}|/B < c|\rho_s|/B_s$ should be held at any altitudes in the gap, as MH04a suggested, where $\rho_s$ denotes the real charge density at the stellar surface. However, the assumption that $\rho_{eff}/B$ is a small positive constant may be too strong, because it is only a sufficient condition of $c|\rho_{eff}|/B < c|\rho_s|/B_s$.

In this chapter, on the other hand, we assume that the magnetic fluxes threading the gap is unchanged, considering that charges freely move along the field lines on the upper (and lower) boundaries. As a result, the gap becomes much thicker than MH04a,b; namely, $\Delta l_{SG} \sim 0.5h_m\Omega_{LC}$, which gives $\varepsilon < 10^{-3}$. Therefore, we can neglect the cross-field motion and justify the constancy of $\rho/B$ in the outer region of the gap, where pair creation is negligible. In the inner magnetosphere, $\rho_{eff}/B$ becomes approximately a negative constant, owing to the discharge of the copiously created pairs. Because of this negativity of $\rho_{eff}/B$, a positive $E_\parallel$ is exerted. For a super-GJ solution, we obtain $j_e + j_{ion} \sim 0.9 > \rho_{eff}/(\Omega B/2\pi)$, which guarantees the equipotentiality of the boundaries. For a sub-GJ solution, a problem may occur regarding the equipotentiality; nevertheless, we are not interested in this kind of solutions.

It is noteworthy that the electric current induced by a negative $E_\parallel$ contradicts with the global current patterns if the gap is located near the last-open field line. (No return current sheets on the last-open field line is supposed in slot gap models.) This situation is illustrated in figure 11 (see also the right panel of fig. 1). The current continuity suggests that a positive $E_\parallel$ should be exerted in the particle accelerator if it is located near the last-open field line.

In short, whether the gap solution becomes MH04a way (with a negative $E_\parallel$ as an out-
ward extension of the polar-cap model) or this-work way (with a positive $E_\parallel$ as an inward extension of the outer-gap model) entirely depends on the transfield thickness and on the resultant $\rho_{\text{eff}}/B$ variation along the field lines. If $\Delta l_{\text{SG}} \sim r_s/10$ holds in the outer magnetosphere, $\rho_{\text{eff}}/B$ could be a small positive constant by the cross-field motion of charges (without pair creation); in this case, the current is slightly sub-GJ with electron emission from the neutron star surface, as MH04a,b suggested. On the other hand, if $\Delta l_{\text{SG}} > \theta_{\text{LC}}/40$ holds in the outer magnetosphere, $\rho_{\text{eff}}/B$ takes a small negative value by the discharge of the created pairs (see fig. 10); in this case, the current is super-GJ with ion emission from the surface, as demonstrated in this chapter. Since no studies have ever successfully constrained the gap transfield thickness, there is room for further investigation on this issue.

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