THE GAUSS–LEGENDRE SKY PIXELIZATION FOR THE CMB POLARIZATION (GLESP-POL) ERRORS DUE TO PIXELIZATION OF THE CMB SKY

ANDREI G. DOROSHKEVICH
Astro Space Center of Lebedev Physical Institute, Profsoyuznaya 84/32, Moscow, Russia
dorr@asc.rssi.ru

OLEG V. VERKHODANOV
Special Astrophysical Observatory, Nizhnij Arkhyz, Karachay-Cherkesia, 369167, Russia
vo@sao.ru

PAVEL D. NASELSKY* and JAISEUNG KIM†
Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
*naselsky@nbi.dk
†jkim@nbi.dk

DMITRY I. NOVIKOV
Imperial College, London, United Kingdom
Astro Space Center of Lebedev Physical Institute, Profsoyuznaya 84/32, Moscow, Russia
d.novikov@imperial.ac.uk

VIKTOR I. TURCHANINOV
Keldysh Institute of Applied Mathematics, Russian Academy of Science, Moscow, 125047, Russia

IGOR D. NOVIKOV
Astro Space Center of Lebedev Physical Institute, Profsoyuznaya 84/32, Moscow, Russia
Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

1053
We present the development of the method for numerical analysis of polarization in the Gauss–Legendre sky pixelization (GLESP) scheme for CMB maps. This incorporation of the polarization transforms in the pixelization scheme GLESP completes the creation of our new method for numerical analysis of CMB maps. A comparison of GLESP and HEALPix calculations is done.

Keywords: Cosmic microwave background; data analysis.

1. Introduction

The analysis of the anisotropy of CMB temperature and polarization is one of the most effective methods for the extraction of the cosmological information and for tests of cosmology and fundamental physics.

The measurements of the anisotropy of CMB temperature performed by the WMAP mission allow us to establish parameters of the cosmological model of the Universe with unprecedented precision.1–4 Increasing sensitivity and angular resolution of the CMB data, including recently available WMAP five-year data,2 ACBAR5 and QUaD,6,7 stimulates significant development and increasing predictability of the corresponding software (see, for instance, CAMB,8 COSMOMC,9 RICO,10 etc.).

During the next decade, after the PLANCK experiment, an investigation of the CMB polarization (including the $B$ mode) will be the focus of the CMB science. Planning the CMBpol mission,11 the B-pol mission,12 etc. requires significant improvement in estimation of the errors of the signal. Partially, uncertainties due to pixelization of the CMB sky and their propogation to the CMB power and the map can be potential sources of the error. For planning high resolution, the CMB experiments most frequently used the HEALPix package13 as well as GLESP,14 ECP,15 and some others, needs to be tested in order to provide exact information about the error bars of the convolution of the $T, E, B$ maps to the corresponding coefficients of the spherical harmonics decomposition not only for the power spectrum, but also for the real and imaginary parts of the coefficients.
The information about the multipole structure of the CMB signal is vital for the low multipoles ($\ell = 2, 3, \ldots$), since a lot of theoretical predictions about the properties of the cosmological model are related directly to the global morphology of the signal. As an example, we would like to mention the widely discussed Bianchi VII$_h$ anisotropic cosmological model,\textsuperscript{16–18} which can mimic the anisotropy of the CMB power at the range of multipoles $\ell \leq 20$, and the Cold Spot\textsuperscript{19} as well. The whole sky decomposition is vital for testing the alignment and planarity of the CMB anisotropy multipoles at $2 \leq \ell \leq 5$, discussed in Ref. 20. The exact information about the phases of the CMB signal is very useful for investigation of statistical anisotropy and non-Gaussianity of the CMB.\textsuperscript{21–24}

This paper is devoted to presentation of a new package, GLESP-pol (see Appendix), and investigates in detail the errors of the standard transition “map to $a_{\ell,m}$” and vice versa for the most frequently used HEALPix 2.11 and the newly released GLESP-pol package. We would like to point out that both these packages reveal some peculiarities of the reconstruction of the coefficients of decomposition, especially for polarization. The major part of the error belongs to the $\ell, m = 0$ and $\ell, m = 2$ modes, when the simplest variants of decomposition were used. For HEALPix 2.11, it is the “zero iteration” key which blocks the correction of the $a_{\ell,m}$ taken from the map. For the GLESP-pol package, the maxima of the $a_{\ell,m}$ errors correspond to the GLESP 1.0 pixelization.\textsuperscript{14,25} However, all these problems can be successfully resolved by implementation of iterations for HEALPix 2.11 (the key “iterative analysis” three or four iterations) and the GLESP-pol pixelization for the polarization.

The outline of the paper is as follows. In Sec. 2, we discuss the difference between the HEALPix and the GLESP-pol scheme of the CMB sky pixelization, focusing on the polarization of the CMB. Section 3 is devoted to investigation of the errors of the “map $\rightarrow a_{\ell,m} \rightarrow$ map” transition for the CMB temperature anisotropy for HEALPix and GLESP-pol. In Sec. 4, we discuss the same issue for the $Q, U$ Stokes parameters and $E$ and $B$ modes of polarization. In the Appendix, the basic relations and description of the GLESP-pol package are presented.

2. Basic Definitions

The temperature and polarization CMB anisotropy can be described in terms of the Stokes parameters $T, Q, U$ through spherical harmonics decomposition $Y_{\ell,m}(\theta, \phi)$, and spin $\pm 2$ spherical harmonics $\pm 2 Y_{\ell,m}(\theta, \phi)$:

\[
T(\theta, \phi) = \sum_{\ell} \sum_{m} a_{\ell,m} Y_{\ell,m}(\theta, \phi),
\]

\[
Q(\theta, \phi) \pm iU(\theta, \phi) = \sum_{\ell} \sum_{m} \pm 2 a_{\ell,m} \pm 2 Y_{\ell,m}(\theta, \phi).
\]

(1)

Here $\theta, \phi$ are the polar and azimuthal angles of the polar system of coordinates, $a_{\ell,m}$ stands for the temperature anisotropy, and the spin coefficients $\pm 2 a_{\ell,m}$ can be decomposed into $E$ and $B$ modes of polarization (see Appendix for details):

\[
\pm 2 a_{\ell,m} = -(a_{\ell,m}^E \pm ia_{\ell,m}^B), \quad a_{\ell,m}^{E,B} = (-1)^m (a_{E,B}^{\ell,-m}).
\]

(2)
The conversion of the $T,Q,U$ signals to the corresponding $a_{\ell,m}$ and $\pm 2a_{\ell,m}$ coefficients is given by the following integrals:

$$0,\pm 2a_{\ell,m} = \int_{-1}^{1} dx \int_{0}^{2\pi} d\phi (T(x,\phi),Q(x,\phi),U(x,\phi))Y_{\ell,m}^*(x,\phi),$$

where the index 0 marks the temperature anisotropy, and $x = \cos \theta$. As is seen from Eq. (3), the mathematical basis of any scheme of the pixelization of the CMB sky is very simple. We need to estimate the integrals in Eq. (3) with very high accuracy, taking into account the properties of discrete representation of the signal on the sphere. However, the modern CMB experiments normally deal with the incomplete sky due to peculiarities of design, scan strategy or implementation of different sorts of masks. In this case the scheme of the pixelization of the sky becomes even more important, since we need to use the pixel domain for estimation of the power spectrum and investigation of the statistical properties of the CMB signal without implementation of the $0,\pm 2a_{\ell,m}$ coefficients. For these purposes the basic idea of the HEALPix package (equal area isolatitude pixelization) is very useful and more advanced in comparison with other pixelization. However, for the whole sky analysis of the $T,Q,U$ signals, the $0,\pm 2a_{\ell,m}$ domain seems to be more optimal, in terms of the CPU timing, as from the scientific point of view. This is why in this paper we propose the GLESP-pol pixelization as a complementary approach to the HEALPix package.

For numerical evaluation of the integrals in Eq. (3), we use the Gaussian quadratures. This approach was proposed by Gauss in 1814, and developed by Christoffel in 1877. As the integral over $x$ in Eq. (3) is an integral over a polynomial of $x$, we can use the equality:

$$\int_{-1}^{1} dx \Delta T(x,\phi)Y_{\ell,m}^*(x,\phi) = \sum_{j=1}^{N} w_j \Delta T(x_j,\phi)Y_{\ell,m}^*(x_j,\phi),$$

where both $\Delta T(x_j,\phi)Y_{\ell,m}^*(x_j,\phi)$ and the proper Gaussian quadrature weighting functions, $w_j = w(x_j)$, are taken at points $x_j$, which are the net of roots of the Legendre polynomial

$$P_N(x_j) = 0.$$  

Here $N$ is the maximal rank of the polynomial under consideration.

It is well known that the equation $P_N(x_j) = 0$ has $N$ number of zeros in the interval $-1 \leq x \leq 1$. For the Gaussian–Legendre method (4), the weighting coefficients are

$$w_j = \frac{2}{1-x_j^2}[P'_{N}(x_j)]^{-2},$$

where the prime denotes a derivative. They can be calculated together with the set of $x_j$ with the “gauleg” code.

The old GLESP (version 1.0) pixelization scheme was defined as follows:

- In the polar direction $x = \cos \theta$, we define $x_j, j = 1,2,\ldots,N$, as the net of roots of Eq. (5).
Each root \( x_j \) determines the position of a ring with \( N_j^\phi \) pixel centers having \( \phi \) coordinates \( \phi_i \).

- All the pixel centers have a nearly equal area.
- Each pixel has a weight \( w_j \) [see Eq. (6)].

This scheme for the temperature anisotropy was realized in the publicly available code presented in www.glesp.nbi.dk, and below we call it “GLESP 1.0,” denoted as grA (grid of equal areas). The new code GLESP-pol is based on the same definition of the roots of the Legendre polynomials, but a different definition of the pixel area (see item 3 above):

- All the rings have the same number of pixels (case grN — grid of equal numbers of pixels in a ring) by default. That is a cylindric projection of the sphere. We have also checked a special case where the number of pixels of a ring has an increment 4 starting from 10 pixels near the poles (case grS — special grid).

The old scheme (GLESP 1.0) where all the pixels have a nearly equal area (grA) is also accessible.

In Fig. 1, we show the differences in pixelization of GLESP 1.0, GLESP-pol (grN), GLESP-pol (grS) and HEALPix. One can see, in the central part of the map, that the properties of GLESP-pol (grN, grS) are the same as for GLESP 1.0, but in the vicinity of the polar cups they are significantly different due to overpixelization. This modification is vital for estimation of the \( a_{\ell,m} \), \( \pm Y_{\ell,m}(\theta, \phi) \) spherical harmonics (see Appendix for details).

### 3. Errors “\( a_{\ell,m} \rightarrow \text{map} \rightarrow a_{\ell,m} \)” transition

This section is devoted to estimation of the error of the \( a_{\ell,m} \) reconstruction by implementing different regimes of the HEALPix 2.11 and the GLESP-pol (grN and grS) packages. The error bars were defined in the following way. Let us take the coefficients \( a_{\ell,m}^{T,E,B} \) for temperature anisotropy \( T \) and \( E \) and \( B \) modes of polarization for the \( \Lambda \)CDM concordance model by implementation of the Monte Carlo simulation of the random Gaussian signal.

For the \( C_{\ell,\ell} \) spectrum, we have taken a value equal to \( 10^{-11} \) just to escape zero in the test calculations.

Then, by implementation of the HEALPix 2.11 and GLESP-pol packages, we use these \( a_{\ell,m}^{T,E,B} \) coefficients to create the map of the signal, keeping for the two packages the same number of pixels. Thus, the HEALPix map \( M_H \) and the GLESP-pol map \( M_G \) are defined as

\[
M_H = H a_{\ell,m}^{T,E,B}, \quad M_G = G a_{\ell,m}^{T,E,B},
\]

\( H \) is the HEALPix projection matrix,

\( G \) is the GLESP-pol projection matrix.

\( a_{\ell,m}^{T,E,B} \) are defined as

\[
a_{\ell,m}^{T,E,B} = \int \int d^2 \ell Y_{\ell,m}(\theta, \phi) Y_{\ell,m}^{*}(\theta, \phi),
\]

\( a_{\ell,m}^{T,E,B} \) are defined as

\[
\sum_{\ell,m} a_{\ell,m}^{T,E,B} a_{\ell,m}^{T,E,B} = 1
\]

\( a_{\ell,m}^{T,E,B} \) is the unique characteristic, which determines all the statistical properties.

---

\( ^a \) Actually, the increment and starting number can be changed optionally and here they are fixed for convenience.

\( ^b \) We have used the Gaussian signal for simplicity. However, only for the Gaussian signal is the power spectrum \( C(\ell) = (2\ell + 1)^{-1} \sum_{m=-\ell}^{\ell} |a_{\ell,m}^{T,E,B}|^2 \) the unique characteristic, which determines all the statistical properties.
Fig. 1. Left column: Molldweide projection of pixelization grids (from top to bottom): (a) the standard pixelization grid of GLESP 1.0, (b) the GLESP-pol rectangular grid with the same number of pixels for each ring (the so-called case grN), (c) the GLESP-pol grid with a pixel number increment 4 (case grS) starting from 10 pixels near poles but not greater than a given resolution in the equator ring, (d) the HEALPix grid. Right column: Corresponding pixelization in the vicinity of the polar cups.
where $H$ and $G$ are the HEALPix and the GLESP-pol operators for the “$a_{\ell,m}$ to map” transition, the index $i$ marks the number of iteration (0 as default, or 1–4), the index $p = 1$ marks the GLESP-pol $grN$ pixelization, and $p = 2$ corresponds to the $grS$ pixelization. Let us define the corresponding transition “map to $a_{\ell,m}$” as $H^{-1}$ for HEALPix and $G^{-1}$ for GLESP-pol:

$$
\begin{align*}
    b_{t,m}^{T,E,B} &= H^{-1}M_H = H^{-1}H a_{t,m}^{T,E,B}, \\
    c_{t,m}^{T,E,B} &= G^{-1}M_G = G^{-1}G a_{t,m}^{T,E,B},
\end{align*}
$$

(8)

where $b_{t,m}^{T,E,B}$ and $c_{t,m}^{T,E,B}$ are now the reconstructed coefficients for HEALPix and GLESP-pol, correspondingly. For the idealistic case, when the “$a_{\ell,m} \rightarrow map \rightarrow a_{\ell,m}$” transition has no error bars, the reconstructed $b_{t,m}^{T,E,B}$ and $c_{t,m}^{T,E,B}$ have to be identical to the input coefficients $a_{t,m}^{T,E,B}$, and $H^{-1}H = HH^{-1} = I$, $G^{-1}G = GG^{-1} = I$, where $I$ is just a unit matrix. In reality, neither the HEALPix nor the GLESP-pol package has a nonzero error of the reconstruction due to window functions of the pixels and computational errors for the spherical harmonics. This means that the corresponding absolute errors for the “$a_{\ell,m} \rightarrow map \rightarrow a_{\ell,m}$” transition can be defined as follows:

$$
\begin{align*}
    R_{H,t,m}^i &= \Re(c_{t,m}^{T,E,B}) - \Re(a_{t,m}^{T,E,B}), \\
    R_{G,t,m}^p &= \Re(c_{t,m}^{T,E,B}) - \Re(a_{t,m}^{T,E,B}), \\
    I_{H,t,m}^i &= \Im(c_{t,m}^{T,E,B}) - \Im(a_{t,m}^{T,E,B}), \\
    I_{G,t,m}^p &= \Im(c_{t,m}^{T,E,B}) - \Im(a_{t,m}^{T,E,B}),
\end{align*}
$$

(9)

where $\Re$ and $\Im$ stand for the real and imaginary parts of the coefficients. Thus, the relative error is given by

$$
\begin{align*}
    r_{H,t,m}^i &= \frac{R_{H,t,m}^i}{\Re(a_{t,m}^{T,E,B})}, \\
    r_{G,t,m}^p &= \frac{R_{G,t,m}^p}{\Re(a_{t,m}^{T,E,B})}, \\
    \gamma_{H,t,m}^i &= \frac{I_{H,t,m}^i}{\Im(a_{t,m}^{T,E,B})}, \\
    \gamma_{G,t,m}^p &= \frac{I_{G,t,m}^p}{\Im(a_{t,m}^{T,E,B})}.
\end{align*}
$$

(10)

Note that the relative errors defined in Eq. (10) are related to the error of the angular power spectrum

$$
\frac{\Delta C_\ell}{C_\ell} = \frac{\sum_m (|g_{\ell,m}|^2 - |a_{\ell,m}|^2)}{\sum_m |a_{\ell,m}|^2} = \frac{\sum_m |a_{\ell,m}|^2 (\delta_{\ell,m} + \delta_{\ell,m}^*)}{\sum_m |a_{\ell,m}|^2},
$$

(11)

where $C_\ell = 1/(2\ell + 1)[|a_0|^2 + 2\sum_{m=1}^{\ell} |a_{\ell,m}|^2]$ is the angular power spectrum, $g_{\ell,m}$ and $a_{\ell,m}$ denote the reconstructed and input multipole coefficients, and $g_{\ell,m} = a_{\ell,m}(1 + \delta_{\ell,m})$. Taking into account that $\delta_{\ell,m} + \delta_{\ell,m}^* = 2\Re(\delta_{\ell,m})$ and $\Re(\delta_{\ell,m}) = |(\Re a_{\ell,m})^2 r_{\ell,m} + (\Im a_{\ell,m})^2 y_{\ell,m}|/|a_{\ell,m}|^2$, where $r_{\ell,m}$ and $y_{\ell,m}$ denote relative errors for real and imaginary parts from Eq. (10), we get

$$
\frac{\Delta C_\ell}{C_\ell} = \frac{2 \sum_m [\Re a_{\ell,m}]^2 r_{\ell,m} + (\Im a_{\ell,m})^2 y_{\ell,m}}{\sum_m |a_{\ell,m}|^2}.
$$

(12)
From Eq. (12) it is clearly seen that the error of reconstruction of the real and imaginary parts of each $\ell, m$ coefficient propagates to the error of the power spectrum through the weighting coefficients

$$w_{\ell,m}^r = \frac{(\Re a_{\ell,m})^2}{\sum_m |a_{\ell,m}|^2}, \quad w_{\ell,m}^\imath = \frac{(\Im a_{\ell,m})^2}{\sum_m |a_{\ell,m}|^2} \quad (13)$$

and formally depends on the power spectrum and the morphology of the input signal.

### 3.1. Errors for HEALPix and GLESP-pol temperature anisotropy

We will start our analysis from estimation of the dynamical range of variations of the temperature anisotropy for $N_{\text{side}} = 1024$. For that we will use the input signal, which corresponds to random Gaussian CMB, and the Internal Linear Combination Map (WILC5) from the LAMBDA archive (http://lambda.gsfc.nasa.gov) and the corresponding K, Ka, Q, V and W total channel maps. In Fig. 2, we plot the diagram

![Diagram](image.png)

Fig. 2. (a) The $A(t)$ versus $A(\text{th})$ diagram [i.e. $|a_{\ell,m}(t)|$ vs. $|a_{\ell,m}(\text{th})|$] for the WILC5 (black dots), K band (blue dots), Ka band (green dots) and V band (red dots). Both of the values $A(t)$ and $A(\text{th})$ are in mK. (b) The same as the left plot, but for replacement of the WILC5 by $A(\text{th})$. (c) The power spectrum $C(\ell)$ mK$^2$ for the WILC5 (red bottom line), for random realization (black line), the K band (solid blue line), the Ka band (solid green line) and the V band (solid red line).
A(i) versus A(th), with A(i) = string(a_\ell,m), where the operator “string” transforms the a_\ell,m coefficients to one-dimensional string |a_{1,0}|, |a_{1,1}|, \ldots for K–W bands, and A(th) is the string for the random Gaussian CMB signal. From this diagram one can see that, for example, to estimate the a_\ell,m coefficients by using the ILC method we should have, at least, the relative error better than 10^{-3}–10^{-4} if the low frequency K-band is included in the analysis. The accuracy of the a_\ell,m coefficients should be about 3–4 orders of magnitude better if we are interested in different sorts of coupling between different multipoles (the so-called non-Gaussianity tool).

To estimate the errors of the “a_\ell,m \to map \to a_\ell,m” transition, let us start from the analysis of the temperature anisotropy for HEALPix 2.11 and GLESP-pol packages. In Fig. 3, we show the maps for the b_\ell,m - a_\ell,m signal, where b_\ell,m is obtained by implementation of the HEALPix 2.11 “zero iteration” key (top left map) and four iterations (top right map). The maps are produced from the differences of the corresponding spherical harmonic coefficients.

One can see that the top left map reveals all the peculiarities of the pixelization, localized at the vicinity of the north and south poles. The top left plot shows the map of differences, but after four iterations. No visible large scale defects can be found. The bottom left and right maps represent the GLESP-pol pixelization without any iterations for the same number of pixels as the top ones. In Fig. 4, we plot the real and imaginary parts of the absolute errors from Eq. (9) for the maps, shown in Fig. 3.
The definition of the function $A(in)$ and $r$ in Fig. 4 is the following: $A(in) = \text{string}(a_{\ell,m})$, where the operator “string” transforms the $a_{\ell,m}$ coefficients into one-dimensional string $a_0, a_1, a_2, \ldots$, and $r = |r^{0}_{H\ell,m}, r^{4}_{H\ell,m}, y^{0}_{H\ell,m}, y^{4}_{H\ell,m}|$ [see Eq. (10)].

Figure 4 clearly show that the HEALPix 2.11 (no iteration) reconstruction is different for the real and imaginary parts of the signal. For the real part (see Fig. 4, top left), the relative error has the secondary zone, localized at $r > 10^{-5}$, while for the imaginary part the major part of the points belongs to $r < 10^{-5}$. Note that even after implementation of four iterations for HEALPix 2.11 and grN, grS modes of GLESP-pol, there exist a very small number ($\sim 10$) of modes with error 1–10%. Thus, the main conclusion is that for $N_{\text{side}} = 1024$ the HEALPix iterations significantly improve the global morphology of the map and the error of reconstruction, which is practically the same for HEALPix and GLESP-pol.

The next question which we would like to discuss is how the error of reconstruction depends on the number of pixels and their size. Both these parameters are determined by the choice of $N_{\text{side}}$ for HEALPix, or by the maximal resolution of the map $\ell_{\text{max}}$ for GLESP-pol. Answering this question in Figs. 5 and 6, we plot...
Fig. 5. The differences of reconstructed and input maps for HEALPix 2.11 for $N_{\text{side}} = 32$ (top pair) and GLESP-pol (bottom pair). The top left plot corresponds to the “zero iteration” key (the color scale is $-10^{-4}, 10^{-4}$ mK); the top right plot is for four iterations (the color scale is $-8 \cdot 10^{-8}, 8 \cdot 10^{-8}$ mK). Bottom left: The GLESP-pol reconstruction for the grN mode (the color scale is $-10^{-9}, 10^{-9}$ mK). The bottom right corresponds to the grS mode (the color scale is $-2.5 \cdot 10^{-8}, 2.5 \cdot 10^{-8}$ mK). The number of pixels for HEALPix 2.11 and for GLESP-pol are practically the same. There is no correction by the window function of the pixels.

It is important to note that as for the high resolution map of differences, shown in Fig. 3, and as for the low resolution map (see Fig. 5) for HEALPix 2.11 with zero iterations the major component of the error is related to the $b_{\ell,m=0}$ mode. These harmonics manifest themselves as horizontal lines parallel to the galactic plane. Two horizontal lines with high amplitude signal along mark the HEALPix zones, where the number of pixels for each equal latitude ring starts to decrease, when $\theta \rightarrow 0$ (the north pole) or $\theta \rightarrow \pi$ (the south pole). However, after four iterations all these peculiarities of the map of errors were significantly suppressed except for two zones around the north and south cups. For the GLESP-pol package with $\ell_{\text{max}} = 32$, the minimal level of errors for the reconstructed map is given by the grN pixelization, when $r \ll 10^{-7}$ for the major part of the pixels, and $r \ll 10^{-5}$ for the grS pixelization. It is important to note that the accuracy of reconstruction of real and imaginary parts of the coefficients of expansion is different for HEALPix and GLESP-pol. As follows from Figs. 4 and 6, HEALPix recovers the imaginary part significantly better than a real one even for the zero-iteration mode. GLESP-pol reconstructs real and imaginary parts with nearly equal errors.
4. The GLESP-pol and the HEALPix Polarization

As mentioned in the Introduction, the GLESP-pol pixelization was designed to assess the problem of accurate reconstruction of the coefficients of spin $\pm 2$ spherical harmonics decomposition (see Appendix for details). Since the spin $\pm 2$ spherical harmonics have a peculiarity, from the computational point of view, behavior in the vicinity of the polar cups, the differences in the GLESP-pol and the HEALPix produce different errors of reconstruction and become more visible, especially for polarization. In Fig. 7, we plot the map of differences between input and output signals, reconstructed by GLESP-pol (grN and grS pixelization).

As one can see from this figure, the reconstruction of $Q$ and $U$ components by the grN and grS pixelization is characterized by very high accuracy, but the grN pixelization looks slightly better. For HEALPix 2.11 the corresponding maps for differences are shown in Fig. 8.

For low resolution pixelization with $N_{\text{side}} = 32$ and $\ell_{\text{max}} = 32$, the corresponding maps are shown in Fig. 9.
Fig. 7. The differences of reconstructed and input maps for GLESP-pol for $\ell_{\text{max}} = 1500$ (the top pair is for $Q$, the Stokes parameter, and the bottom pair is for $U$). The top left plot corresponds to the grN pixelization (the color scale is $-3 \cdot 10^{-10}, 3 \cdot 10^{-10}$ mK); the top right plot is for the grS pixelization (the color scale is $-10^{-9}, 10^{-9}$ mK). Bottom left: The GLESP-pol reconstruction of $Q$ for the grN mode (the color scale is $-2.5 \cdot 10^{-10}, 2.5 \cdot 10^{-10}$ mK). The bottom right corresponds to the grS mode (the color scale is $-10^{-9}, 10^{-9}$ mK).

Fig. 8. The differences of reconstructed and input maps for HEALPix 2.11 for $N_{\text{side}} = 1024$ (the top pair corresponds to $Q$ components and the bottom pair is for $U$). The top left plot corresponds to the “zero iteration” key (the color scale is $-10^{-8}, 10^{-8}$ mK); the top right plot is for four iterations (the color scale is $10^{-10}, 10^{-10}$ mK). Bottom left: The GLESP-pol reconstruction for the grS mode (the color scale is $10^{-8}, 10^{-8}$ mK). The bottom right corresponds to the grN mode (the color scale is $-10^{-10}, 10^{-10}$ mK). The number of pixels for the given HEALPix 2.11 and GLESP-pol maps are practically the same. There is no correction by the window function of the pixels.
Fig. 9. The differences of reconstructed and input maps for HEALPix 2.11 for $N_{\text{side}} = 32$ (the top pair corresponds to $Q$ components and the bottom pair is for $U$). The top left plot corresponds to the “zero iteration” key (the color scale is $-10^{-6}, 10^{-6}$ mK); the top right plot is for four iterations (the color scale is $-3 \cdot 10^{-10}, 3 \cdot 10^{-10}$ mK). Bottom left: The $U$ component for the zero iterations (the color scale is $-10^{-6}, 10^{-6}$ mK). The bottom right corresponds to four iterations (the color scale is $-3 \cdot 10^{-10}, 3 \cdot 10^{-10}$ mK).

Fig. 10. The errors of reconstruction for GLESP-pol ($Q$ for the top pair and $U$ for the bottom pair; maps prepared for $\ell \leq 60$). The top left plot corresponds to the grN pixelization (the color scale is $-2 \cdot 10^{-11}, 2 \cdot 10^{-11}$ mK); the top right plot is for grN (the color scale is $-10^{-8}, 10^{-8}$ mK). Bottom left: The GLESP-pol reconstruction for the grS mode (the color scale is $-2 \cdot 10^{-11}, 2 \cdot 10^{-11}$ mK. The bottom right corresponds to the grS mode (the color scale is $-10^{-8}, 10^{-8}$ mK).
Sky Pixelization for the CMB Polarization (GLESP-Pol) Errors

Fig. 11. The errors of reconstruction of $\Re e E_{\ell,m}$ and $\Im m E_{\ell,m}$. The top row corresponds to HEALPix 2.11 ($N_{\text{side}} = 1024$). From left to right $E$ mode with zero iterations, $E$ mode with four iterations, $B$ mode with zero iterations, $B$ mode with four iterations. The bottom row is the same, but for the GLESP-pol gN and gS modes and $\ell_{\text{max}} = 1500$. Black dots correspond to the real part of $\ell, m$ modes, and red dots are for the imaginary part.
Thus, one can see that implementation of the four iterations for the HEALPix package gives us practically the same result as for GLESP-pol even for the low resolution maps (see Fig. 10). In Fig. 11 we show the diagram, similar to Fig. 4 but for the $E$ and $B$ components of polarization.

One can see that the GLESP-pol mode grN gives us practically the same errors as the HEALPix 2.11 four-iteration mode. However, the CPU time for GLESP-pol is about four times smaller than for HEALPix 2.11, due to the absence of iterations.

Finally, we demonstrate absolute relative accuracy for power spectra $[\Delta C(\ell)/C(\ell)]$ for GLESP 2.0 (grS and grN) and HEALPix 2.11 (zero- and four-iteration modes), shown in Fig. 12. As one can see, the realtive accuracy for restoration of the power spectrum of temperature anisotropy in GLESP is approximately the same for the two types, grS and grN, and the HEALPix four-iteration mode.

![Graphs showing relative accuracy of power spectra](image)

Fig. 12. The relative accuracy of $C(\ell)$ spectrum restoration for temperature anisotropy (a), $E$ polarization (b) and $B$ polarization (c). The grS GLESP type is shown with the black line, the grN type is shown in red, the HEALpix calculations of $C(\ell)$ of zero iteration mode are plotted in green, and of four iteration in blue. (Color online.)
gives better accuracy at low multipoles and reaches the GLESP one at higher. For polarization, we have approximately the same accuracy for the GLESP grN and HEALPix four-iteration modes.

5. Conclusions

We have presented the GLESP-pol package, which incorporates calculations of polarization on the sphere into the CMB analysis package based on the Gauss–Legendre sky pixelization. We developed corresponding software for data processing.

According to our numerical calculations, the described scheme for polarization preserves the same level of precision as GLESP 1.0 can provide for the temperature anisotropy.

The code continues to be developed now and different definitions of polarization modes are planned to be included. GLESP-pol is open for new approaches and could be implemented for extension of the present code, such as the fast spin-weighted harmonics calculation. A completely new algorithm for the fast spherical harmonics transform has been proposed recently by Tygert. It is worth mentioning that this algorithm can be applied only for GLESP pixelization of the sky and is effective as \(O(A \times \ell^2 \log_2 \ell)\) operations instead of the usual \(O(\ell^3)\). Unfortunately, a huge prefactor \(A\) makes this algorithm only about three times faster than the existing HEALPix and GLESP for \(\ell = 2048\).

The most important part of our investigation is that the HEALPix 2.11 package provides the same accuracy as the GLESP-pol package only by implementation of four iterations. The zero-iteration mode of HEALPix 2.11 can provide significant error, for the coefficients of expansion and should be used with caution.

Acknowledgments

The authors are very grateful to the anonymous referee and Radek Stompor for useful remarks on the paper, and to Per Rex Christensen for very fruitful discussions and for checking the GLESP procedures. This paper was supported in part by the RFBR grants 08-02-00090, 08-02-00159, 09-02-00298, by S.S. 2469.2008.2 and by the FNU grants 272-06-0417, 272-07-0528, 21-04-0355. O. V. V. is grateful to Dmitry Zimin’s Foundation “Dynasty” for support of the work.

Appendix

A.1. Description of polarization

A.1.1. Basic relations for the flat space

The polarization is characterized by the symmetric traceless matrix \(T\), composed of two Stokes parameters, \(Q(x)\) and \(U(x)\):

\[
T = \begin{pmatrix}
Q & U \\
U & -Q
\end{pmatrix}.
\]  

(A.1)
The functions $Q$ and $U$ depend upon the coordinate frame, and components of the tensor $T^j_i$ obey the corresponding tensor transformation law:

$$\tilde{T}^j_i = O^k_i O^j_k T^k_i,$$

(A.2)

where the coordinate transformation is given by $\tilde{x}^i = O^i_k x^k$. In the particular case under rotation of the coordinate system with the rotation angle $\phi$,

$$O = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix},$$

(A.3)

the functions $Q$ and $U$ are transformed as

$$Q' = Q \cos^2 \phi + U \sin 2\phi,$$

$$U' = -Q \sin^2 \phi + U \cos 2\phi.$$

(A.4)

The intensity $I^2 = Q^2 + U^2 = \frac{1}{2} T^k_k T^l_l$ does not depend upon the coordinate system.

A.1.2. Basic relations on the sphere

An arbitrary traceless symmetric tensor can be presented in terms of scalar “electrical” ($E$) and pseudoscalar “magnetic” ($B$) potentials as

$$T_{ij} = \left( E_{;ij} - \frac{1}{2} \delta_{ij} E_{;k}^k + \frac{1}{2} (\epsilon_{ij} B_{;k}^k + \epsilon_{jk} B_{;i}^k) \right).$$

(A.5)

where $\epsilon_{ij}$ is the completely antisymmetric tensor ($\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = -\sin \theta$) and a semicolon denotes covariant differentiation on the two-dimensional sphere with the metric

$$dl^2 = d\theta^2 + \sin^2 \theta d\varphi^2.$$

(A.6)

Here $\theta$ and $\varphi$ are the polar and azimuthal angles. In this case we get, instead of (A.1),

$$T^1_1 = -T^2_2 = Q \quad T^1_2 = \sin \theta U, \quad T^2_1 = \frac{U}{\sin \theta},$$

(A.7)

and again we have

$$I^2 = 0.5 T^k_k T^l_l = Q^2 + U^2.$$

(A.8)

The Stokes parameters $Q$ and $U$ are linked with the potentials $E$ and $B$ as

$$Q = D_1 E - D_2 B, \quad U = D_2 E + D_1 B,$$

(A.9)

and the operators $D_1$ and $D_2$ are

$$D_1 = \frac{1}{2} \left( \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right)$$

$$= \frac{1}{2} \left( (1 - x^2) \frac{\partial^2}{\partial x^2} - \frac{1}{1 - x^2} \frac{\partial^2}{\partial \varphi^2} \right),$$

(A.10)
A.1.3. Spherical harmonics

Following Refs. 14 and 25, we use the representation of scalar (“electrical”), $E$, and pseudoscalar (“magnetic”), $B$, polarization potentials identical to (1):

$$E = \frac{1}{\sqrt{2\pi}} \sum_{\ell=2}^{\ell_{\text{max}}} \left( a_{\ell0} f_{\ell}^0(x) + 2 \sum_{m=1}^{\ell} f_{\ell m}^x(x) (e_{\ell m}^c \cos(m\varphi) - e_{\ell m}^s \sin(m\varphi)) \right),$$

(A.12)

$$B = \frac{1}{\sqrt{2\pi}} \sum_{\ell=2}^{\ell_{\text{max}}} \left( b_{\ell0} f_{\ell}^0(x) + 2 \sum_{m=1}^{\ell} f_{\ell m}^b(x) (b_{\ell m}^c \cos(m\varphi) - b_{\ell m}^s \sin(m\varphi)) \right),$$

(A.13)

$$f_{\ell}^m = \sqrt{\frac{(\ell+0.5)(\ell-m)!}{(\ell+m)!}} B_{\ell}^m, \quad 0 \leq m \leq \ell \leq l_{\text{max}}.$$  

(A.14)

Here $B_{\ell}^m(x)$ and $f_{\ell}^m(x)$ are the associated Legendre functions (ordinary and normalized) and $e_{\ell m}$ and $b_{\ell m}$ are the coefficients of decomposition characterizing properties of polarization.

$$e_{\ell m}^c = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} dx \int_{0}^{2\pi} d\varphi E(x,\varphi) f_{\ell}^m(x) \cos m\varphi,$$

(A.15)

$$e_{\ell m}^s = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} dx \int_{0}^{2\pi} d\varphi E(x,\varphi) f_{\ell}^m(x) \sin m\varphi.$$  

(A.16)

Similar expressions for the “magnetic” mode can be obtained by replacing $e_{\ell m}^c$ and $e_{\ell m}^s$ with $b_{\ell m}^c$ and $b_{\ell m}^s$ and $E$ with $B$.

As in the previous package, the functions $f_{\ell}^m(x)$ are found recursively:

$$f_{\ell}^m(x) = x \sqrt{\frac{4\ell^2 - 1}{\ell^2 - m^2}} f_{\ell-1}^m - \sqrt{\frac{2\ell + 1}{2\ell - 3}} \frac{(\ell+1)(\ell-1)^2 - m^2}{\ell^2 - m^2} f_{\ell-2}^m,$$

(A.17)

or

$$f_{\ell}^m = -\frac{2(m-1)}{\sqrt{\ell^2 - m^2 + \ell + m}} \sqrt{\frac{\ell + 2 - m}{\ell + 1 - m}} \sqrt{\frac{\ell + m - 1}{\ell + m}} f_{\ell-2}^m,$$

(A.18)

$$f_{m}^m = (-1)^m \sqrt{\frac{(2m+1)!!}{2(2m)!!}} (1 - x^2)^{m/2}, \quad f_{m+1}^m = x \sqrt{2m + 3} f_{m}^m.$$  

(A.19)
The relation (A.17) starts with \( f^m_\ell \) and \( f^{m+1}_\ell \) [Eqs. (A.19)] and generates the functions \( f^m_\ell \) for all \( \ell \geq m \). The relation (A.18) starts with \( f^m_\ell \) and \( f^0_\ell \) and generates the functions \( f^m_\ell \) for all \( 1 \leq m \leq \ell \). In this case, the function \( f^m_\ell \) must be found with the relation (A.17).

It is easy to see that

\[
D_1 E = F^m_\ell \left[ c^\ell_m \cos(m \varphi) - e^\ast_m \sin(m \varphi) \right],
\]

\[
D_2 E = \Phi^m_\ell \left[ c^\ell_m \sin(m \varphi) + e^\ast_m \cos(m \varphi) \right],
\]

where the functions \( F^m_\ell \) and \( \Phi^m_\ell \) are expressed through the normalized Legendre functions (A.17) and (A.18):

\[
M_\ell F^m_\ell = \frac{2\ell+1}{2\ell-1} \frac{\ell f^m_{\ell-1}}{1-x^2} + \frac{(m^2 - \ell) f^m_\ell}{1-x^2} - \frac{\ell(\ell-1)}{2} f^m_\ell,
\]

\[
M_\ell \Phi^m_\ell = \frac{m}{1-x^2} \left[ \sqrt{\frac{2\ell+1}{2\ell-1} (\ell^2 - m^2)} f^m_{\ell-1} - (\ell-1) x f^m_\ell \right],
\]

or in the other way:

\[
M_\ell F^m_\ell = \frac{m-1}{1-x^2} m f^m_\ell - \frac{\ell^2 + \ell - 2m}{2} f^m_\ell - \sqrt{\frac{\ell^2 - m^2 + \ell - m}{1-x^2}} x f^{m+1}_{\ell+1},
\]

\[
M_\ell \Phi^m_\ell = -m \sqrt{\frac{(\ell + m + 1)(\ell - m)}{1-x^2}} f^{m+1}_{\ell+1} + (m-1) \frac{x}{1-x^2} f^m_\ell.
\]

Here

\[
M^2_\ell = 0.25(\ell + 2)(\ell + 1)(\ell - 1),
\]

and the functions \( F^m_\ell \) and \( \Phi^m_\ell \) are normalized by the condition

\[
\int_{-1}^{1} (F^m_\ell + \Phi^m_\ell)^2 dx = \int_{-1}^{1} [(F^m_\ell)^2 + (\Phi^m_\ell)^2] dx = 1.
\]

In particular,

\[
f^m_m = \frac{m+1}{2M_m} f^m_m, \quad f^m_{m+1} = m x f^m_m \frac{2m+3}{M_{m+1}} \left[ \frac{m-1}{1-x^2} - \frac{m+1}{2} \right],
\]

\[
\Phi^m_m = -\frac{m(m-1)}{M_m(1-x)} x f^m_m, \quad \Phi^m_{m+1} = m f^m_m \frac{2m+3}{M_{m+1}} \frac{1-3mx^2}{1-x^2}.
\]

Combining (A.9), (A.10), (A.12) and (A.13), we get for the functions \( Q \) and \( U \)

\[
Q^c_\ell(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\varphi Q \cos m \varphi = \sum_{\ell=2}^{\ell_{\text{max}}} (F^c_\ell(x) c^\ell_m \Phi^c_\ell(x) \Phi^c_\ell_m),
\]

\[
Q^s_\ell(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\varphi Q \sin m \varphi = \sum_{\ell=2}^{\ell_{\text{max}}} (-F^s_\ell(x) c^\ell_m \Phi^s_\ell(x) \Phi^s_\ell_m),
\]
These functions satisfy the self-adjoint equation
\[ U_m^m(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\varphi \cos m\varphi = \sum_{\ell=2}^{\ell_{\text{max}}} (\Phi_{\ell}^m(x) c_{\ell m}^{\ell} + F_{\ell}^m(x) b_{\ell m}^{\ell}), \quad (A.32) \]
\[ U_s^m(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\varphi \sin m\varphi = \sum_{\ell=2}^{\ell_{\text{max}}} (\Phi_{\ell}^m(x) c_{\ell m}^{\ell} - F_{\ell}^m(x) b_{\ell m}^{\ell}). \quad (A.33) \]

A.1.4. Spin-weight functions

The functions \( F_{\ell}^m \) and \( \Phi_{\ell}^m \) do not form an orthogonal basis and to perform the further decomposition of polarization we need to use the normalized spin-weight spherical functions, which can be defined as follows:

\[ \lambda_+^{\ell,m} = F_{\ell}^m + \Phi_{\ell}^m, \quad \lambda_-^{\ell,m} = F_{\ell}^m - \Phi_{\ell}^m, \quad (A.34) \]
\[ \lambda_+^{\ell,m}(x) = (-1)^{m+\ell}\lambda_-^{\ell,m}(-x), \quad \lambda_-^{\ell,m}(x) = (-1)^{m+\ell}\lambda_+^{\ell,m}(-x). \quad (A.35) \]

These functions satisfy the self-adjoint equation
\[ \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \lambda_{\ell,m} \right] - \frac{4 + m^2 + 4mx}{1-x^2} \lambda_{\ell,m} + \ell(\ell+1)\lambda_{\ell,m} = 0, \quad (A.36) \]
and so they are orthogonal for a given \( m \) (see e.g. Refs. 30 and 31 and references therein). Some of them can be written directly:

\[ \lambda_+^{m,m} = A_{m,m}(1-x)^2(1-x^2) \frac{\partial}{\partial x} = F_{m}^m + \Phi_{m}^m, \quad (A.37) \]
\[ \lambda_+^{m+1,m} = A_{m+1,m}(1-x)^2(1-x^2) \frac{\partial}{\partial x} [2 + (m+1)x] = F_{m+1}^m + \Phi_{m+1}^m, \quad (A.38) \]
\[ A_{m,m}^2 = \frac{m(m-1)(2m+1)!!}{(m+2)(2m+2)!!}, \quad A_{m+1,m}^2 = \frac{2m+3}{(m+3)(m-1)} A_{m,m}^2, \quad (A.39) \]

and using these relations the functions \( \lambda_{\ell,m} \), with \( \ell > m+1 \), can be evaluated recursively:

\[ \lambda_+^{\ell,m} = \left( x + \frac{2m}{\ell(\ell-1)} \right) C_{\ell,m} \lambda_{\ell-1,m} - C_{\ell,m} \lambda_{\ell-2,m}, \quad (A.40) \]
\[ C_{\ell,m} = \sqrt{\frac{\ell^2(4\ell^2-1)}{(\ell^2-m^2)(\ell^2-4)}}, \quad (A.41) \]
\[ \int_{-1}^{1} dx [F_{\ell}^{m+1}(x) F_{\ell}^{m+1}(x) + \Phi_{\ell}^{m+1}(x) \Phi_{\ell}^{m+1}(x)] = \delta_{\ell}, \quad (A.42) \]
\[ \int_{-1}^{1} dx [F_{\ell}^{m+1}(x) \Phi_{\ell}^{m+1}(x) + \Phi_{\ell}^{m+1}(x) F_{\ell}^{m+1}(x)] = 0. \quad (A.43) \]
This means that coefficients for the decomposition of $Q$ and $U$ can be found from (A.30)–(A.33) as follows:

$$
e_{\ell m}^c = \int_{-1}^{1} dx \left[ \Phi_{\ell m}^c(x) U_{\ell m}^c(x) + F_{\ell m}^c(x) Q_{\ell m}^c(x) \right],$$  
(A.44)

$$
e_{\ell m}^s = \int_{-1}^{1} dx \left[ \Phi_{\ell m}^s(x) U_{\ell m}^s(x) - F_{\ell m}^s(x) Q_{\ell m}^s(x) \right],$$  
(A.45)

$$
b_{\ell m}^c = \int_{-1}^{1} dx \left[ F_{\ell m}^c(x) U_{\ell m}^c(x) - \Phi_{\ell m}^c(x) Q_{\ell m}^c(x) \right],$$  
(A.46)

$$
b_{\ell m}^s = \int_{-1}^{1} dx \left[ -F_{\ell m}^s(x) U_{\ell m}^s(x) - \Phi_{\ell m}^s(x) Q_{\ell m}^s(x) \right].$$  
(A.47)

For numerical analysis of the polarization maps together with the temperature fluctuations, it is convenient to use the relations (A.14)–(A.25) and (A.44)–(A.47) instead of the recursive relations (A.40). The inverse problem, i.e. the construction of the polarization maps from the coefficients $e_{\ell m}$ and $b_{\ell m}$, can also be solved using the relations (A.30)–(A.33).

A.1.5. Spectra of the anisotropy and polarization

Four power spectra can be introduced for the temperature, electrical and magnetic Stokes parameters:

$$C_T^\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2,$$  
(A.48)

$$C_E^\ell = \frac{1}{2\ell + 1} \sum_{m=0}^{\ell} [(e_{\ell m}^c)^2 + (e_{\ell m}^s)^2],$$  
(A.49)

$$C_B^\ell = \frac{1}{2\ell + 1} \sum_{m=0}^{\ell} [(b_{\ell m}^c)^2 + (b_{\ell m}^s)^2],$$  
(A.50)

$$C_{TE}^\ell = \frac{1}{2\ell + 1} \sum_{m=0}^{\ell} [a_{\ell m}^c e_{\ell m}^c + a_{\ell m}^s e_{\ell m}^s],$$  
(A.51)

When averaged over the sky, the mean square temperature anisotropy is

$$\langle \Delta T^2 \rangle = \sum_{\ell=2}^{\infty} \frac{2\ell + 1}{4\pi} C_T^\ell,$$  
(A.52)

where $T_0$ is the temperature of the CMB. The mean square of polarization is

$$\langle I^2 \rangle = \frac{1}{2} \langle Q^2 + U^2 \rangle = \frac{1}{2} \langle (E_z^2) + (B_z^2) \rangle,$$  
(A.53)
Sky Pixelization for the CMB Polarization (GLESP-Pol) Errors

where

\[
\langle E_\ell^2 \rangle = \sum_{\ell=2}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell}^{EE},
\]

(A.54)

\[
\langle B_\ell^2 \rangle = \sum_{\ell=2}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell}^{BB}.
\]

(A.55)

More details can be found in, for example, Refs. 32–34, 27.

A.2. Incorporating the polarization into the GLESP code

To realize algorithms described above, special procedures have been created. We have developed this code in parallel in the two algorithmic languages, GNU C and FORTAN-77. The procedures have been designed both like subroutines and package utilities implemented in the GLESP pixelization scheme.

Two commands of GLESP-pol should be used for calculation of polarization, namely `polmap` and `polalm`. `Polmap` calculates \( Q \) and \( U \) polarization maps by coefficients of \( E \) and \( B \) polarization modes. `Polalm` calculates coefficients of \( E \) and \( B \) polarization modes by \( Q \) and \( U \) polarization maps, respectively. \( E \) and \( B \) coefficients are recorded as \( a_{\ell m} \) format files, and \( Q \) and \( U \) maps are recorded like GLESP maps. Both procedures can be used for temperature–spherical harmonics transformations too. They are included in the GLESP package, version 2.0. The programs use the fast Fourier transform FFTW-3.2.1.35

The program `difmap` is developed for standard polarization transforms to calculate a polarization angle \( PA = \frac{1}{2} \arctan(\frac{U}{Q}) \) and an intensity \( I = \sqrt{Q^2 + U^2} \). Both values can be plotted with the GLESP drawing procedure `f2fig`.

A.3. Accuracy restrictions

Both temperature and polarization maps and spectra processed by the GLESP package are determined mainly by four parameters. First of all, it is the maximal number of harmonics under consideration, \( \ell_{\text{max}} \) [Eq. (A.12)]. The second one is the number of rings used for map presentation, \( N_{\theta} \). By definition, we must have \( N_{\theta} \geq 2\ell_{\text{max}} \). The third one is the number of pixels for each ring of the map, \( N_{i\phi}(\theta_i) \). If we would like to use the same angular resolution in the azimuthal and polar directions, then, evidently, we must take \( N_{i\phi}(\pi/2) \approx 2N_{\theta} \) and \( N_{i\phi}(\theta_i) \) decreases progressively away from the equator. The fourth one is the Nyquist parameter, \( N_y \), which regulates the precision achieved by calculations.

The precision depends upon the choice \( N_{\theta} \geq 2\ell_{\text{max}} \) and the Nyquist parameter, \( N_y \). Thus, the orthogonality of spherical harmonics \( Y_{\ell m} \) on the sphere is achieved only if the number of pixels for each ring is at least two times larger than the number of harmonics we use in our analysis, \( N_{i\phi}(\theta_i) \geq 2m, 0 \leq m \leq \ell \) [Nyquist restriction, \( N_{i\phi}(\theta_i) \geq m/N_y \geq 2m, N_y \leq 0.5 \)]. Since we try to keep the same pixel area for
different latitudes, this condition is not completely satisfied anymore, because rings with number of pixels $N^i_{\phi}(\theta_i) \leq 2m$ drop out from any calculations.

Fortunately, the influence of these restrictions on the precision achieved is moderate because the contribution of polar areas is quite small for almost all harmonics for the following reason:

- Spin-weighted harmonics satisfy $\pm 2 Y_{\ell m} \sim \sin^{m-2}(\theta)$ for $m > 2$. This means that harmonics with high $m$ are weighted negligibly everywhere apart from the equator, where the number of pixels is enough for orthogonality.
- Harmonics with small $m \leq m_{\text{min}} = N_{\phi} \times N_{\min}$ are orthogonal.
- Therefore, it can be expected that the harmonics with $m \sim m_{\text{min}}$ are the most dangerous. For such harmonics, rings near the poles have the number of pixels smaller than $2 \times m$ and these parts of the sphere spoil the picture. However, the weight of these parts of the sphere is proportional to $\sin^{m-2}(\theta)$ and tends to be zero for $\theta \to 0, \pi$.

To accelerate the computations, we retain in the calculations only larger terms restricted by the conditions $|f^m_{\ell}(x)| \geq \varepsilon$ and $|F^m_{\ell}|, |\Phi^m_{\ell}| \geq \varepsilon$, where $\varepsilon$ depends upon $\ell$ and the required precision, and so is determined in practice. Our tests show that the choice $\varepsilon = \varepsilon_1$ for $\ell \leq 500$ and $\varepsilon = \varepsilon_2$ for $\ell \geq 500$ allows one to moderately decrease the calculation time and weakly change the precision achieved.

We can conclude that for a given $\ell_{\text{max}}$ reasonable precision can be achieved for the parameters

$$N_{\theta} = (2.5-3)\ell_{\text{max}}, \quad N_{\phi} = 2 \times N_{\theta}, \quad N_{\min} = 9-11, \quad N_{\phi} = 0.5, \quad (A.56)$$

and

$$\varepsilon_1 \approx 10^{-10}, \quad \varepsilon_2 \approx 10^{-4}. \quad (A.57)$$

Evidently, the precision achieved increases for larger $N_{\theta}$ and $N_{\phi}$; however, it is accompanied by corresponding fast growth of the calculation time. Perhaps, for calculations of polarization better results can be achieved with repixelization of the map in accordance with the net of roots of the spin-weight functions $\lambda^\pm_{\ell m}$ [Eqs. (A.34)] instead of the Legendre polynomials (5) and the corresponding choice of the weighting coefficients (6).

### A.4. Data format

In developing the GLESP package for polarization calculation, we should change the format of data representation. We introduce two types of the format describing $a_{\ell m}$ coefficients and maps.

In the first case, we can use the standard $a_{\ell m}$ coefficients data, which contain an index describing the number of $\ell$ and $m$ modes corresponding to the HEALPix, real and imaginary parts of $a_{\ell m}$. These three parameters are described by three-field records of the FITS binary table.\(^{36}\) Map data are described by the three-field
binary table FITS format containing a vector of $x_i = \cos \theta$ positions, a vector of numbers of pixels for each layer $N_{i\phi}$, and a set of temperature values in each pixel recorded by layers from the North Pole. All these data description formats are used separately for each type and polarization mode data, i.e. maps for temperature anisotropy, $Q$ and $U$ data are contained in single files, and $a_{\ell m}$ coefficients are stored in single files too.

The second type of data representation format is similar to the HEALPix one. In this case, data are unified in three extensions containing maps with anisotropy, $Q$ and $U$ polarization data or $a_{\ell m}$ coefficients of temperature expansions and $E$ and $B$ modes, respectively. Files with coefficients in GLESP and in HEALPix have the same format. Each of three FITS extensions of GLESP files with maps contains the three fields described above.

So, employing two formats of data representation, a user can easily select a path of data processing including or excluding any type of polarization or anisotropy data.

References